EXERCISES FOR THE COURSE “ALGEBRAIC GEOMETRY” (SOSE 2023) SESSION 3

What to do with these exercises

Try to solve them, of course. Understanding when is the right moment for leaving an exercise unfinished is part of the game. If you are stuck, either try to find a different angle to tackle the problem, or leave the exercise for the moment. You can come back later, or chew the problem during the day.

We will discuss together your efforts on Tuesday. The best thing to do would be to prepare a bunch of solutions/approaches to the problems below.

Exercises

Exercise 1. Show that every regular morphism of the quasi-projective variety $\mathbb{P}^n \smallsetminus (H_i \cap H_j)$ is constant, where $H_i := \{x_i \neq 0\}$ and $H_j := \{x_j \neq 0\}$. What about regular morphisms of $\mathbb{P}^n$?

Exercise 2. Consider the morphism $\mathbb{P}^1 \to \mathbb{P}^3$ given by

$$(x_0, x_1) \mapsto (x_0^3, x_0^2 x_1, x_0 x_1^2, x_1^3)$$

The image of this morphism is a *twisted cubic*, which we call C.

(1) Show that the C is a projective variety, and find the generators the ideal $I(C)$ of homogeneous polynomials that vanish on C.
(2) Show that given four points $p_1, \dots, p_4$ on C, the smallest projective subspace of $\mathbb{P}^3$ that contains $p_1, \dots, p_4$ is $\mathbb{P}^3$ itself.
(3) Show that given seven points $p_1 \dots, p_7$ on C, the intersection of the quadrics (i.e. zero loci of homogeneous polynomials of degree two) that vanish on $\{p_1, \dots, p_7\}$ coincide with C.

Exercise 3. Let $H \subset \mathbb{P}^n$ be a projective subspace, i.e. we have

$$H = \{f(x_0, \dots, x_n) = 0\}$$

with f is a homogeneous polynomial of degree one.

(1) Show that $H \simeq \mathbb{P}^{n-1}$.
(2) Let p be a point in the complement of H. Show that there is a well defined, regular morphism of quasi-projective varieties $\phi : \mathbb{P}^n \smallsetminus p \to H$ that sends every point q to the intersection point of the line $\overline{pq}$ with H.
(3) Let $H = \{x_2 = 0\}$ be a projective subspace of $\mathbb{P}^3$, and consider the induced regular morphism $\phi : \mathbb{P}^3 \smallsetminus (0,0,1,0) \to \mathbb{P}^2$. Compute $I(\phi(C))$, where C is the twisted cubic of Exercise 2.

Exercise 4. Fix n and d positive natural numbers, and let $\rho : \mathbb{P}^n \to \mathbb{P}^N$ for $N = \binom{n+d}{n} - 1$ be the morphism defined as

$$(x_0, \dots, x_n) \longmapsto (M_1(x_0, \dots, x_n), M_2(x_0, \dots, x_n), \dots, M_N(x_0, \dots, x_n))$$

where the $M_i(x_0, \dots, x_n)$ are the monomials of degree d in the variables $x_0, \dots, x_n$. Show that $\rho(\mathbb{P}^n)$ is a projective variety, and compute the ideal of homogeneous polynomials that vanish on $\rho(\mathbb{P}^n)$.