EXERCISES FOR THE COURSE
"ALGEBRAIC GEOMETRY" (SOSE 2023)
SESSION 4

What to do with these exercises

Try to solve them, of course. Understanding when is the right moment for leaving an exercise unfinished is part of the game. If you are stuck, either try to find a different angle to tackle the problem, or leave the exercise for the moment. You can come back later, or chew the problem during the day.

We will discuss together your efforts on Tuesday. The best thing to do would be to prepare a bunch of solutions/approaches to the problems below.

Exercises

Exercise 1. Let $H \subset \mathbb{P}^n$ be a projective subspace, i.e. we have

$$H = \{f(x_0, \dots, x_n) = 0\}$$

with f is a homogeneous polynomial of degree one.

(1) Show that $H \simeq \mathbb{P}^{n-1}$.
(2) Let p be a point in the complement of H. Show that there is a well defined, regular morphism of quasi-projective varieties $\phi : \mathbb{P}^n \smallsetminus p \to H$ that sends every point q to the intersection point of the line $\overline{pq}$ with H.
(3) Let $H = \{x_2 = 0\}$ be a projective subspace of $\mathbb{P}^3$, and consider the induced regular morphism $\phi : \mathbb{P}^3 \smallsetminus (0, 0, 1, 0) \to \mathbb{P}^2$. Compute $I(\phi(C))$, where C is the twisted cubic of Exercise 2.

Exercise 2. Let $\Gamma = \{p_1, p_2, \dots, p_d\}$ be a finite set of distinct points in general position in $\mathbb{P}^n$. Being *in general position* means the following: given any subset of points $S = \{p_{i_1}, \dots, p_{i_r}\} \subset \Gamma$ with $r \leq n + 1$, the smallest projective subspace containing S is isomorphic to $\mathbb{P}^{s-1}$ (the smallest projective subspace containing a set S is called *the span of* S).

(1) Assume $d = 2n$. Let q be a point such that every quadric form that vanishes on the points of Γ vanishes also on q. Show that q belongs to $\Lambda_1 \cup \Lambda_2$, where Λ_1 is the span of $\{p_1, \dots, p_n\}$ and Λ_2 is the span of $\{p_{n+1}, \dots, p_{2n}\}$.
(2) Show that if q belongs to Λ_1 and it's not equal to p_i for $i = 2, \dots, n$, then it does not belong to the span of $\{p_2, \dots, p_n, p_{n+i}\}$ for $1 \leq i \leq n$.
(3) Deduce that q belongs to the span of $\{p_1, p_{n+1}, p_{n+2}, \dots, p_{2n}\} \smallsetminus \{p_{n+i}\}$ for any $1 \leq i \leq n$. Conclude that $q = p_1$. In this way we proved that any finite set of cardinality $d \leq 2n$ can be described as $Z(I)$ with I generated by quadrics.
(4) Use the same argument used in the points above and induction to prove that for $d \leq kn$ with $k \geq 2$, any finite set Γ of cardinality d can be described as $Z(I)$ with I generated by forms of degree $\leq k$.

Exercise 3. Let C be the image of the morphism $\mathbb{P}^1 \to \mathbb{P}^n$ given by

$$(x_0, x_1) \longmapsto (x_0^n, x_0^{n-1}x_1, \dots, x_0 x_1^{n-1}, x_1^n).$$

The curve C is called a *rational normal curve*. Prove that given $kn + 1$ distinct points on C, any form of degree k that vanishes on these points vanishes also on

C. Compare this result with the statement proved in the last point of the previous exercise.

Exercise 4. Fix n and d positive natural numbers, and let $\rho : \mathbb{P}^n \to \mathbb{P}^N$ for $N = \binom{n+d}{n} - 1$ be the morphism defined as

$$(x_0, \ldots, x_n) \longmapsto (M_1(x_0, \ldots, x_n), M_2(x_0, \ldots, x_n), \ldots, M_N(x_0, \ldots, x_n))$$

where the $M_i(x_0, \ldots, x_n)$ are the monomials of degree d in the variables $x_0, \ldots, x_n$. Show that $\rho(\mathbb{P}^n)$ is a projective variety, and compute the ideal of homogeneous polynomials that vanish on $\rho(\mathbb{P}^n)$.