EXERCISES FOR THE COURSE “ALGEBRAIC GEOMETRY” (SOSE 2023) SESSION 5

What to do with these exercises

Try to solve them, of course. Understanding when is the right moment for leaving an exercise unfinished is part of the game. If you are stuck, either try to find a different angle to tackle the problem, or leave the exercise for the moment. You can come back later, or chew the problem during the day.

We will discuss together your efforts on Tuesday. The best thing to do would be to prepare a bunch of solutions/approaches to the problems below.

Exercises

Exercise 1. Let $V = k^{\oplus 2}$ be a vector space over k of rank two, with standard basis $\{v_0, v_1\}$.

(1) Show that the dual space $V^\vee$ has a basis given by the formal derivations $\frac{\partial}{\partial v_0}$ and $\frac{\partial}{\partial v_1}$. Deduce that $V^\vee$ can be identified with the vector space of linear derivations on V.
(2) Given a vector $f \in V$, consider the induced homomorphism $\phi_f : V^\vee \to \mathrm{Sym}^{d-1} V$ that sends a derivation ∂ to $\phi_f(\partial) = \partial(f)$. Show that if $f = (a_0 v_0 + a_1 v_1)^d$ then ϕ_f has rank ≤ 1.
(3) Prove that if φ_f has rank ≤ 1 then $f = (a_0 v_0 + a_1 v_1)^d$.
(4) Use the points above to compute the homogeneous ideal associated to the rational normal curve $C \subset \mathbb{P}^d$ (i.e. the curve given by the image of $\mathbb{P}^1 \to \mathbb{P}^d$, $(x_0, x_1) \mapsto (x_0^d, x_0^{d-1} x_1, \ldots, x_1^d)$).

Exercise 2. Fix n and d positive natural numbers, and let $\rho : \mathbb{P}^n \to \mathbb{P}^N$ for $N = \binom{n+d}{n} - 1$ be the morphism defined as

$$(x_0, \ldots, x_n) \longmapsto (M_1(x_0, \ldots, x_n), M_2(x_0, \ldots, x_n), \ldots, M_N(x_0, \ldots, x_n))$$

where the $M_i(x_0, \ldots, x_n)$ are the monomials of degree d in the variables $x_0, \ldots, x_n$. Show that $\rho(\mathbb{P}^n)$ is a projective variety, and compute the ideal of homogeneous polynomials that vanish on $\rho(\mathbb{P}^n)$. (Hint: use a strategy similar to the one used in the previous exercise)