EXERCISES FOR THE COURSE "ALGEBRAIC GEOMETRY" (SOSE 2023) SESSION 6

What to do with these exercises

Try to solve them, of course. Understanding when is the right moment for leaving an exercise unfinished is part of the game. If you are stuck, either try to find a different angle to tackle the problem, or leave the exercise for the moment. You can come back later, or chew the problem during the day.

We will discuss together your efforts on Tuesday. The best thing to do would be to prepare a bunch of solutions/approaches to the problems below.

Exercises

In what follows κ denotes an algebraically closed field.

Exercise 1. Fix $\alpha_0, \dots, \alpha_d \in \kappa$ and $\beta_0, \dots, \beta_d \in \kappa$ so that for every $i \neq j$, the points (α_i, β_i) and (α_j, β_j) are distinct in $\mathbb{P}^1$. Let $f = \prod_{i=0}^{d}(\alpha_i x_0 - \beta_i x_1)$ be a form of degree $d+1$, and set $h_i = f/(\alpha_i x_0 - \beta_i x_1)$.

(1) Show that $h_0, \dots h_d$ form a basis for the vector space of forms of degree d in two variables.

(2) Show that the image of $\nu_d : (x_0, x_1) \mapsto (h_0, \dots, h_d)$ is a rational normal curve in $\mathbb{P}^d$ that passes through the coordinate points $p_0 = (1, 0, \dots, 0)$, $p_1 = (0, 1, 0, \dots, 0)$,..., $p_d = (0, \dots, 0, 1)$. Assume $\alpha_i, \beta_i \neq 0$ for every i: compute the image of $(0, 1)$ and $(1, 0)$.

(3) Show that any rational normal curve passing through the coordinate points can be written in this way for some choice of α_i and β_i.

(4) Deduce that given $d+3$ points in general position in $\mathbb{P}^d$, there exists a unique rational normal curve passing through these points.

Exercise 2. Let $V = \kappa^{\oplus n+1}$ be a vector space generated by $v_0, \dots, v_n$. Let $f \in \mathrm{Sym}^d V$ be a form of degree d in $v_0, \dots, v_n$.

(1) Consider the homomorphism $\phi_f : V \to \mathrm{Sym}^{d-1} V$ given by $(a_0, \dots, a_n) \mapsto \sum_{i=0}^{n} a_i \frac{\partial}{\partial v_i}(f)$, where $\frac{\partial}{\partial v_i}$ denote the partial derivatives with respect to v_i. Show that if $f = \ell^d$ for some linear form ℓ, than the rank of ϕ_f is ≤ 1. Show that also the viceversa holds.

(2) Deduce from the previous point a set of equations whose zero locus is the set $S \subset \mathrm{Sym}^d V$ formed by all those forms f such that $f = \ell^d$ for some linear form ℓ.

(3) By looking at the projectivization $\mathbb{P}(\mathrm{Sym}^d V) \simeq \mathbb{P}^{\binom{n+d}{n}-1}$, consider $Y = S \smallsetminus \{0\}/\sim$ as a subset of $\mathbb{P}^{\binom{n+d}{n}-1}$. Show that Y is equal (up to a projective linear transformation) to the image of the Veronese embedding

$$\nu_{n,d} : \mathbb{P}^n \longrightarrow \mathbb{P}^{\binom{n+d}{n}-1}, \quad (x_0, \dots, x_n) \longmapsto (m_0, \dots, m_{\binom{n+d}{n}-1}),$$

where the m_i are monomials that form a basis for the space of forms of degree d in $n+1$ variables.

(4) Compute generators for the homogeneous ideal associated to the image of the Veronese embedding.