EXERCISES FOR THE COURSE "ALGEBRAIC GEOMETRY" (SOSE 2023) SESSION 8

What to do with these exercises

Try to solve them, of course. Understanding when is the right moment for leaving an exercise unfinished is part of the game. If you are stuck, either try to find a different angle to tackle the problem, or leave the exercise for the moment. You can come back later, or chew the problem during the day.

We will discuss together your efforts on Tuesday. The best thing to do would be to prepare a bunch of solutions/approaches to the problems below.

Exercises

In what follows we work over an algebraically closed field κ.

Exercise 1. Let $\nu_2 : \mathbb{P}^2 \to \mathbb{P}^5$ be the Veronese embedding defined in the previous exercises, and let Σ be the image of ν_2. We call Σ the Veronese surface.

(1) Find equations for the Veronese surface, using the exercises from the previous sessions if you like.
(2) Let $Y \subset \mathbb{P}^2$ be the Fermat cubic of equation $x_0^3 + x_1^3 + x_2^3 = 0$. Show that $\nu_2(Y)$ can be written as an intersection of nine quadrics.
(3) The previous point holds in higher generality: use the general Veronese embedding to show that any projective variety can be written as an intersection of quadrics.

Exercise 2. In what follows, given a ring of polynomials R, we denote R_d the R-module of homogeneous polynomials of degree d (together with 0).

(1) Let A be a κ-algebra which is an integral domain, and let f, $g \in A[x, y]$ be homogeneous polynomials (forms) of degree respectively d and e. Consider the morphism
$$\phi : A[x, y]_{e-1} \times A[x, y]_{d-1} \longrightarrow A[x, y]_{e+d-1}$$
that sends a pair (p, q) to $fp+gq$. Show that f and g have a common divisor of positive degree if and only if ϕ has not maximal rank. Deduce that there is a polynomial $\rho(a_0, \ldots, a_d, b_0, \ldots, b_e)$ which vanishes if and only if $f = \sum a_i x^i y^{d-i}$ and $g = \sum b_i x^i y^{e-i}$ have a common divisor of positive degree. From now on we use the notation $\rho(f, g)$ instead of $\rho(a_0, \ldots, a_d, b_0, \ldots, b_e)$.
(2) Show that given forms $f_1, \ldots, f_r \in \kappa[x, y]$, they have a common zero if and only if the polynomial $\rho(f_r, u_1 f_1 + \ldots + u_{r-1} f_{r-1}) = 0$ in $\kappa[u_1, \ldots, u_{r-1}]$.
(3) Let $\pi : \mathbb{P}^n \smallsetminus q \to \mathbb{P}^{n-1}$ be the projection from a point, and let $X \subset \mathbb{P}^n$ be a projective variety not containing q. Use the previous points to show that $\pi(X)$ is closed.
(4) Prove that for any quasi-projective variety V, the projection morphism $V \times \mathbb{P}^n \to V$ is closed. (Hint: it blows down to showing that the fibers are non-empty, which is equivalent to show that the image via a certain projection is non-empty. Iterate this process.)