## Lecture 2: moduli of stable maps

**Disclaimer**: these are *very* rough notes. Do not expect them to be exhaustive and/or detailed. There notes are intended more as a roadmap rather than a manual. Any comment is always welcome!

Guiding questions:

- Old enumerative problems concerning plane rational curves
- How to compactify the moduli space of these rational plane curves of fixed degree?

2.1. **Enumerative problems.** One of the motivation to study moduli spaces was that they permit to solve enumerative problems. Let us give an example of enumerative problem:

*How many rational plane curves of degree $d$ pass through $m$ prescribed points in general position?*

Here is a list of answers in some very simple cases:

- $d=1$, $m=1$: the set of lines passing through a fixed point forms a $\mathbb{P}^1$ (in ancient times they used to say $\infty : 1$ solutions).
- $d=1$, $m=2$: just one line passes through two points in general position (i.e. not equal).
- $d=2$, $m=1$: the set of conics passing through one point forms a $\mathbb{P}^4$ (because passing through a point imposes one linear condition on the moduli space of conics $\simeq \mathbb{P}^5$).
- $d=2$, $m=2$: the set of conics passing through two points in general position form a $\mathbb{P}^3$.
- $d=2$, $m=5$: there is only one conic passing through five points in general position.
- $d=3$, $m=1$: the set of rational cubics passing through a fixed point is a variety of dimension 7 (the moduli space of rational cubics is equal to the discriminant hypersurface in the moduli space of cubics, which is a $\mathbb{P}^9$. Passing through a point impose one linear condition).
- $d=3$, $m=8$: ???

In general, we may wonder when to expect that the answer to the question above is a finite number. Let $V_d^0$ be the set of rational plane curves of degree $d$, regarded as a subset of the moduli space of degree $d$ plane curves, which is isomorphic to $\mathbb{P}^{(d+1)(d+2)/2-1}$.

We can upgrade this subset to the level of a locally closed subscheme. This can be done as follows: consider the projective space

$$\mathbb{P}(H^0(\mathbb{P}^2, \mathcal{O}(d))^{\oplus 3})$$

A point $[s_0 : s_1 : s_2]$ of this variety can be seen as the rational morphism $\mathbb{P}^1 \dashrightarrow \mathbb{P}^2$ defined as $[X : Y] \mapsto [s_0(X,Y) : s_1(X,Y) : s_2(X,Y)]$.

Consider the subset $W(2,d)^*$ where the induced rational morphism is regular (i.e. the linear system $\{s_0, s_1, s_2\}$ is base-point free) and it is an immersion. One can show that $W(2,d)^*$ has the structure of an open subscheme. Take the trivial family of projective lines $\mathbb{P}^1 \times W(2,d)^*$: there is an immersion of this family into $\mathbb{P}^2 \times W(2,d)^*$ whose image is family of degree $d$ plane curves. The fact that $\mathbb{P}^{(d+1)(d+2)/2-1}$ is a fine moduli space for degree $d$ plane curves implies that we have a morphism $W(2,d)^* \to \mathbb{P}^{(d+1)(d+2)/2-1}$.

The image of this morphism is $V_d^0$, and it makes $W(2,d)^*$ into a principal $PGL_2$-bundle over $V_d^0$: in this way this latter set inherits a scheme structure, and we are also able to compute its dimension: it is equal to $3(d+1)-1$ (dimension of $W(2,d)^*$) minus 3 (dimension of $PGL_2$), hence $V_d^0$ has dimension $3d-1$.

*Remark* 2.1. There is another way for doing this dimensional computation: a degree $d$ rational curve has arithmetic genus $(d-1)(d-2)/2$, as any other degree $d$ plane curve (this count follows, for instance, from the adjunction formula).

Its normalization will be a smooth genus 0 curve, from which we deduce that the arithmetic genus of the original curve must also be equal to $(0-1)+\delta+1=\delta$, where $\delta$ is the number of nodes.

We deduce that $\delta=(d-1)(d-2)/2$: asking for a curve to have a node is a codimension one condition on the moduli space $\mathbb{P}^{(d+1)(d+2)/2-1}$, hence the locus of degree $d$ curves with $(d-1)(d-2)/2$ nodes must have codimension $(d-1)(d-2)/2$, i.e. dimension $3d-1$.

The request for a degree $d$ planar curve to pass through a fixed point imposes one linear condition on the moduli space $\mathbb{P}^{(d+1)(d+2)/2-1}$, i.e. the moduli of such curves is a hyperplane $H$ in this projective space.

The set of degree $d$ rational curves passing through that point is equal to $V_d^0 \cap H$.

Therefore, if we impose $3d-1$ points, the corresponding moduli space will be given by the intersection of $V_d^0$ with $3d-1$ hyperplanes in general position, hence a finite number of points.

**Definition 2.2.** We define $N_d$ as the number of degree $d$ plane rational curves passing through $3d-1$ fixed points in general position.

So, here it is a question whose answer in the general case baffled several generations of mathematicians:

$$N_d = ?$$

A possible approach to give an answer is the following: let $\overline{V}_d^0$ be the closure of $V_d^0$ in $\mathbb{P}^{(d+1)(d+2)/2-1}$. By construction this is a closed subvariety (i.e. reduced, irreducible) of dimension $3d-1$. We can take the intersection product (either in the Chow ring or in the homology ring) of $[\overline{V}_d^0]$ with $[H]^{3d-1}$ (where $[H]$ is the hyperplane section). The result will be the class of a bunch of points, which we can count (dualizing the whole thing, this is equivalent to integrate a cocycle of top dimension on $\overline{V}_d^0$).

If these cycles intersect transversely, and the intersection is not supported on the boundary of $\overline{V}_d^0$, the number we would obtain at the end should be equal to $N_d$. This happens to be the case, so that we have reduced ourselves to compute the degree of $\overline{V}_d^0$ in $\mathbb{P}^{(d+1)(d+2)/2-1}$. Unfortunately, this is still a pretty tough question, basically because the boundary of $\overline{V}_d^0$ is hard to understand.

Kontsevich took a different path: in order to solve this problem he compactified the moduli space $V_d^0$ in a different way (we definitely need a proper scheme because otherwise we can't integrate). The boundary of this new compactification has a recursive structure, which makes its study way more approachable, and eventually enabled Kontsevich to produce the following amazing formula.

**Theorem 2.3.**

$$N_d + \sum \binom{3d-4}{3d_A-1} d_A^2 N_{d_A} \cdot N_{d_B} \cdot d_A d_b = \sum \binom{3d-4}{3d_A-2} d_A N_{d_A} \cdot d_B N_{d_B} \cdot d_A d_B$$

The proof of this formula will be sketched in the third lecture.

2.2. **Moduli of stable maps.** We start by defining the moduli functor of smooth $n$-marked maps of genus 0:

$$\mathcal{M}_{0,n}(\mathbb{P}^2, d) : S \longmapsto \left\{ \begin{array}{c} \pi : C \to S \text{ families of smooth curves of genus } 0 \\ \text{with } n \text{ pairwise disjoint sections } \sigma_i : S \to C \\ \text{and a morphism } \mu : C \to \mathbb{P}^2 \text{ such that} \\ \text{for each geometric point } \overline{s} \\ \text{the morphism } \mu_{\overline{s}} : C_{\overline{s}} \to \mathbb{P}^2 \text{ has degree } d \end{array} \right\} \Big/ \simeq$$

where a morphism $\mu : C \to \mathbb{P}^2$ has degree $d$ if $\mu_*[C] = d[H]$.

Define $W(2,d)$ as the subset in $\mathbb{P}(H^0(\mathbb{P}^2, \mathcal{O}(d))^{\oplus 3})$ of base-point free rational morphisms $\mathbb{P}^1 \dashrightarrow \mathbb{P}^2$, i.e. regular morphisms. One can show that this is an open subscheme, and it is a fine moduli spaces for maps of this type.

**Proposition 2.4.** *For $n \geq 3$ the moduli problem $\mathcal{M}_{0,n}(\mathbb{P}^2, d)$ admits a fine moduli space, which is isomorphic to $M_{0,n} \times W(2,d)$.*

For $n < 3$, we do not have a fine moduli space representing $\mathcal{M}_{0,n}(\mathbb{P}^2, d)$, but only what is called a *coarse moduli space.* We take the chance here to introduce this additional piece of moduli theory terminology.

**Definition 2.5.** A functor $\mathcal{F} : (Sch)^{op} \to (Set)$ is *coarsely represented* by a scheme $F$ if there exists a natural transformation

$$\mathcal{F} \longrightarrow F(-)$$

such that:

(1) For every field $k$, it induces a bijection between $\mathcal{F}(\overline{k})$ and $F(\overline{k})$.
(2) For every scheme $X$ and natural transformation $f : \mathcal{F} \to X(-)$, there exits a morphism $F \to X$ such that the composition $\mathcal{F} \to F(-) \to X(-)$ is equal to $f$.

You can think of the coarse moduli space of a non-finely representable moduli functor as the scheme whose functor of points best approximates the moduli functor.

The coarse moduli space will be smooth at those points parametrizing objects having trivial automorphism group, and it will look like the quotient of a smooth scheme by the action of a finite group at those points that parametrize objects with non-trivial automorphism group (basically, the presence of automorphisms is what prevent us from having a fine moduli space). In particular, the coarse moduli space will be an orbifold.

**Proposition 2.6.** *There exists a coarse moduli space $M_{0,n}(\mathbb{P}^2, d)$ for the moduli problem $\mathcal{M}_{0,n}(\mathbb{P}^2, d)$. When $n \geq 3$, the scheme $M_{0,n}(\mathbb{P}^2, d)$ a fine moduli space.*

For $n = 0$, the coarse moduli space $M_{0,0}(\mathbb{P}^2, d)$ can be constructed by taking the space $W(2,d)$ and quotiening out by the natural action of $PGL_2$ defined as:

$$[s_0 : s_1 : s_2] \longmapsto [s_0(A^{-1}(X,Y)) : s_1(A^{-1}(X,Y)) : s_2(A^{-1}(X,Y))]$$

The group $PGL_2$ acts freely on the open subscheme $W(2,d)^*$ of birational maps, and the (smooth) quotient $M_{0,0}(\mathbb{P}^2, d)^*$ is a fine moduli space for the subfunctor $\mathcal{M}_{0,0}(\mathbb{P}^2, d)^*$ of smooth *and birational* maps.

Using geometric invariant theory we can form a quotient $W(2,d)/PGL_2$, which is the desired coarse moduli space $M_{0,0}(\mathbb{P}^2, d)$.

A big issue with $M_{0,n}(\mathbb{P}^2, d)$ is that it is not proper. This can already be seen from the functor, as the next example shows:

**Example 2.7.** Consider the degree 2 morphism $\mathbb{P}^1 \times \mathbb{A}^1 \setminus \{0\} \to \mathbb{P}^2$ given by:

$$([X : Y], t) \longmapsto ([X^2 : tY^2 : XY])$$

The image of the fibre over a point $t_0$ in $\mathbb{A}^1\backslash\{0\}$ is the conic of equation $UV - tW^2 = 0$.

If we try to extend the morphism to the whole family $\mathbb{P}^1 \times \mathbb{A}^1$, we see that there is no way to define in a coherent way the morphism at the point $([0:1], 0)$.

On the other hand, by looking at the image, we see a family of smooth conics degenerating to the rank 2 conic $UV = 0$, so we would like to have a pair of $\mathbb{P}^1$ as our central fibre, so that the extended morphism would have degree 1 on each component.

This can actually be done by performing a blow-up at the incriminated point: in this way we end up with a well defined morphism from a family of *semi-stable* curves to $\mathbb{P}^2$ of degree $d$.

The example above suggests what kind of additional families we need to consider in orderto compactify (in a modular way) the functor $\mathcal{M}_{0,n}(\mathbb{P}^2, d)$.

**Definition 2.8.** A stable $n$-marked, degree $d$ and genus 0 map $\mu : C \to \mathbb{P}^2$ is a morphism such that:

(1) The curve $C$ is a tree of rational curves (at most nodal singularities, dual graph contains no cycle) such that every irreducible component which get contracted by $\mu$ has at least 3 special points (either markings or nodes).
(2) $\mu_*[C] = d[H]$.

**Definition 2.9.** A family of stable $n$-marked, degree $d$ maps of genus 0 over $S$ is a diagram

$$\begin{array}{ccc} X & \xrightarrow{\mu} & \mathbb{P}^2 \\ {\scriptstyle \pi}\downarrow & & \\ S & & \end{array}$$

together with $n$ sections $\sigma_i : S \to X$ such that $\pi$ is flat and proper, and for each geometric point $\overline{s}$ in $S$, the morphism $\mu_{\overline{s}} : X_{\overline{s}} \to \mathbb{P}^2$ is a stable $n$-marked degree $d$ morphism of genus 0.

*Remark* 2.10. The domain of a stable map is not necessarily a stable $n$-marked curve of genus 0. Indeed, the condition of having more than 2 special points is requested only for those components that get contracted by the map.

Morally, the condition of having more than 2 special points on each component of a stable curve is the unique way to assure us that the automorphism group of the curve is not infinite.

In the case of maps, the automorphism group of a component which do not get contracted to a point is automatically finite, because every such automorphism must act as the identity on the image $\mu(C) \subset \mathbb{P}^2$.

We are ready to introduce the moduli functor of stable $n$-marked degree $d$ maps of genus zero:

$$\overline{\mathcal{M}}_{0,n} : S \longmapsto \left\{ \begin{matrix} \text{families of stable } n \text{ marked, degree } d \\ \text{morphisms of genus 0 over } S \end{matrix} \right\} \Big/ \simeq$$

**Proposition 2.11.** *There exists a coarse moduli space $\overline{M}_{0,n}(\mathbb{P}^2, d)$ for the moduli functor above, which is a projective normal irreducible scheme having only finite quotient singularities.*

The Proposition above comes somehow with no surprise: indeed, one can verifies that the *stack* $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^2, d)$ is a proper, smooth Deligne-Mumford stack, and by some general machinery almost every stack of this type admits a coarse moduli space satifying the properties listed in the Proposition.

*Remark* 2.12. In the particular case $n = 0$, $d = 2$, the scheme $\overline{M}_{0,0}(\mathbb{P}^2, 2)$ is actually a smooth, fine moduli space. It is isomorphic to the scheme of *complete conics.* The latter can be realized, for instance, as the blow-up of the moduli space of plane conics $\mathbb{P}^5$ along the codimension 3 closed subscheme of rank 1 conics.