Lecture 3: the boundary of $\overline{M}_{0,n}(\mathbb{P}^r, d)$

Disclaimer: these are *very* rough notes. Do not expect them to be exhaustive and/or detailed. There notes are intended more as a roadmap rather than a manual. Any comment is always welcome!

Guiding questions:

- What is the structure of the boundary of $\overline{M}_{0,n}$ and $\overline{M}_{0,n}(\mathbb{P}^r, d)$?
- How do these different moduli spaces relate one with each other?

3.1. **The boundary of $\overline{M}_{0,n}$.** Recall that in the first lecture we constructed the proper, fine moduli scheme $\overline{M}_{0,n}$ of stable n-marked curves of genus 0. A natural object to study is the boundary $\partial\overline{M}_{0,n} := \overline{M}_{0,n} \setminus M_{0,n}$: what is its codimension? What are the irreducible components? And so on.

Example 3.1. Some particular cases:

(1) For $n = 4$, the boundary $\partial\overline{M}_{0,4}$ is equal to the three points $\{0\}$, $\{1\}$ and $\{\infty\}$. Therefore, the boundary has codimension 1 and it has 3 irreducible components.

Given a smooth 4-marked curve of genus zero, we can always assume that the first three markings correspond to 0, 1 and ∞.

Then we see that the stable curve corresponding to the boundary point $\{0\}$ in $\overline{M}_{0,4}$ is the one obtained as stable limit of a $\mathbb{P}^1$ where the fourth marking σ_4 is approaching the first one σ_1: this limit is a pair of $\mathbb{P}^1$ glued at one point such that the markings σ_1 and σ_4 are on one irreducible component, and the markings σ_2 and σ_3 are on the other one.

It is easy to see that the other boundary points correspond to a different recombination of the markings on the irreducible components.

(2) For $n = 5$, we have $\overline{M}_{0,5} \simeq \overline{U}_{0,4}$, where the latter is the universal family over $\overline{M}_{0,4}$, and $M_{0,5} \simeq (\mathbb{P}^1 \setminus \{0, 1, \infty\})^{\times 2} \setminus \Delta$, where Δ is the diagonal. Recall that $\overline{U}_{0,4}$ was constructed blowing up $\mathbb{P}^1 \times \mathbb{P}^1$ (see first lecture).

We deduce that the boundary has codimension 1 and its components are the proper transform $\{p\} \times \mathbb{P}^1$ and $\mathbb{P}^1 \times \{p\}$ for p in $\{0, 1, \infty\}$, the diagonal Δ and the three exceptional divisors, i.e. the boundary has 10 irreducible components.

What are the stable marked curves corresponding to the generic points of each component? To answer this question, observe that the two coordinates in $M_{0,5} \subset \overline{M}_{0,5}$ amount to the last two markings σ_4 and σ_5.

The generic point of $\{0\} \times \mathbb{P}^1$ is obtained as a limit of the family where the first coordinate of $M_{0,5} \subset \overline{M}_{0,5}$ converge to 0, i.e. σ_4 converges to σ_1: when they collide, a new component spring out, and σ_1, σ_4 are sent to this new component.

We deduce that the generic point associated to this boundary divisor is a curve with two components, with σ_1, σ_4 on one side and σ_2, σ_3 and σ_5 on the other.

A similar description holds for the boundary components $\{1\} \times \mathbb{P}^1$ and $\{\infty\} \times \mathbb{P}^1$.

Consider now the divisor $\mathbb{P}^1 \times \{0\}$: the curve associated to the generic point of this divisor can be obtained by degenerating a smooth marked curve where σ_5 converges to σ_1 (just take the generic point of $\overline{M}_{0,4}$, consider its fibre in $\overline{M}_{0,5}$ and you obtain such a degeneration).

The limit of this family is easy to describe: when σ_1 and σ_5 collide, a new component spring out and σ_1 and σ_5 are sent to this new component.

In other terms, the curve associated to the generic point of $\mathbb{P}^1 \times \{0\}$ has two components with markings σ_1 and σ_5 on one side, and the remaining ones on the other side.

A similar description holds for the boundary components $\mathbb{P}^1 \times \{1\}$ and $\mathbb{P}^1 \times \{\infty\}$.

The generic point of the diagonal Δ can be obtained by making σ_4 and σ_5 coincide: the stable limit has two components with markings $\{\sigma_1, \sigma_2, \sigma_3\}$ on one side and $\{\sigma_4, \sigma_5\}$ on the other.

Finally, consider the exceptional divisor obtained after blowing-up at 0: its generic point can be obtained by making the generic point (σ_4, σ_5) of $M_{0,5}$ degenerate along a generic direction towards the point $(0, 0)$. Therefore, the corresponding curve will be the stable limit of the family of smooth 5-marked curves where both σ_4 and σ_5 are converging to σ_1: this limit will be the union of two $\mathbb{P}^1$ with markings $\{\sigma_2, \sigma_3\}$ one one component and the remaining ones on the other.

From the above examples we can easily argue how the story goes in the general case.

Proposition 3.2. *The generic point of an irreducible component of the boundary $\partial\overline{M}_{0,n}$ is a curve with two irreducible components and n-markings distributed on the two components in such a way to make the curve stable.*

Each boundary component has codimension 1 *and it is smooth.*

Given a partition $A \cup B$ of $\{1, 2, \ldots, n\}$ with $|A|, |B| \geq 2$, we call $D(A|B)$ the boundary component of $\overline{M}_{0,n}$ whose generic point correspond to the curve with the markings σ_a for a in A on one component, and the markings σ_b for b in B on the other component.

Sketch of proof. The fact that each of this component has codimension 1 can be seen as follows: we are free to put $|A| - 2$ markings on the first component (as we can always assume that two of them together with the node belong to the set $\{0, 1, \infty\}$).

Similarly, we are free to decide where to put $|B| - 2$ markings on the other component: hence the dimension of this locus is $|A| + |B| - 4 = n - 4$, hence it has codimesion 1 in $\overline{M}_{0,n}$.

Another way of seeing this is by means of the *gluing morphism*: this is the morphism

$$\overline{M}_{|A|+1} \times \overline{M}_{|B|+1} \longrightarrow \overline{M}_{0,n}$$

which sends a pair of points $([C, \underline{\sigma}], [C' \underline{\sigma}')$ (here $\underline{\sigma}$ denotes the collection of all markings) to the curve obtained by gluing the marking $\sigma_{|A|+1}$ of C with the marking $\sigma'_{|B|+1}$ of C', with sections given by $\sigma_1, \ldots, \sigma_{|A|}, \sigma'_1, \ldots, \sigma_{|B|+1}$ (to show that this morphism actually exists and it is well defined, one can either exploit the fact that all the schemes involved are fine moduli spaces, and construct appropriate families of curves that induce the desired morphism, or rather work directly with the moduli functors/stacks and construct a natural transformation of functors).

The gluing morphism constructed above gives an isomorphism of the domain with the boundary divisor $D(A|B)$. From this we immediately see that the a priori only-just-a-closed-subscheme $D(A|B)$ is an irreducible, smooth and codimension 1 closed subscheme. □

From this we deduce that $\partial\overline{M}_{0,2n}$ has

$$\frac{1}{2}\binom{2n}{n} + \binom{2n}{n-1} + \cdots + \binom{2n}{2}$$

components, and $\partial\overline{M}_{0,2n+1}$ has

$$\binom{2n+1}{n}+\binom{2n+1}{n-1}+\cdots+\binom{2n}{2}$$

irreducible components.

Another important morphism between moduli spaces of stable marked curves is the *forgetful morphism*: this is a morphism

$$\overline{M}_{0,n}\longrightarrow\overline{M}_{0,n-m}$$

for $n-m\geq 3$ that sends a curve C with markings $\sigma_1,\ldots,\sigma_n$ to the same curve C but with markings $\sigma_1,\ldots,\sigma_{n-m}$.

It can be shown that the fibres of this morphism are reduced. In particular, the preimage of $D(A'|B')\subset\overline{M}_{0,n-m}$ in $\overline{M}_{0,n}$ is formed by the union of the boundary divisors $D(A|B)$ where $A'\subset A$ and $B'\subset B$.

The following case

$$\overline{M}_{0,n}\longrightarrow\overline{M}_{0,4}\simeq\mathbb{P}^1$$

is quite relevant because, due to the linear equivalence relation $D(A'|B')\sim D(A''|B'')$ for every partition $A'\cup B'=A''\cup B''=\{1,2,3,4\}$ (this is a simple consequence of the fact that any two points in $\mathbb{P}^1$ are linearly equivalent), we deduce that also their preimages in $\overline{M}_{0,n}$ must be linearly equivalent.

Lemma 3.3. *Let i,j,k,l be distinct elements in $\{1,2,\ldots,n\}$. Then in Mnbar we have:*

$$D(A\cup\{i,j\}|B\cup\{k,l\})\sim D(A\cup\{i,k\}|B\cup\{j,l\})$$

for any partition $A\cup B=\{1,\ldots,n\}\setminus\{i,j,k,l\}$.

3.2. **The boundary of $\overline{M}_{0,n}(\mathbb{P}^2,d)$.** We move now from $\overline{M}_{0,n}$ to $\overline{M}_{0,n}(\mathbb{P}^r,d)$. Recall that, even for $n\geq 3$, the scheme $\overline{M}_{0,n}(\mathbb{P}^r,d)$ is not smooth and it is not a fine moduli space. Nevertheless, it is still irreducible, it is normal and it has at most finite quotient singularities. It contains an open subscheme $\overline{M}_{0,n}(\mathbb{P}^r,d)^*$ which is smooth and that represents the subfunctor of *birational* stable maps. Consequently, over this scheme there exists a universal family of birational stable maps.

We will investigate the boundary of $\overline{M}_{0,n}(\mathbb{P}^r,d)$ using our accumulated knowledge of $\partial\overline{M}_{0,n}$ together with the *forget-the-map morphism*

$$F:\overline{M}_{0,n}(\mathbb{P}^r,d)\longrightarrow\overline{M}_{0,n}$$

that sends a stable map $(C\to\mathbb{P}^r)$ to the marked stable curve C^{st}: by C^{st} here we mean the canonical model of C, i.e. the image of the morphism induced by the linear series $|\omega_C(\sigma_1+\cdots+\sigma_n)|$ (in simpler terms, this is the morphism that contracts the unstable components of C [stability of the map $C\to S$ does not imply stability of C!]).

Remark 3.4. To construct the forget-the-map morphism, there are basically two ways: the first one amounts to show that locally such a morphism exists, using the local structure of $\overline{M}_{0,n}(\mathbb{P}^r,d)$. We haven't pursued the investigation of the local structure of $\overline{M}_{0,n}(\mathbb{P}^r,d)$ in these notes so far, so it does not make much sense to spend much words on this method.

The other possibility is to define a natural transformation

$$\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d)\longrightarrow\overline{\mathcal{M}}_{0,n}$$

which only remembers the family of curves out of the family of maps, after possibly stabilizing it.

Then we can compose this morphism of stacks with the morphism $\overline{\mathcal{M}}_{0,n}\to\overline{M}_{0,n}$ (which exists because $\overline{M}_{0,n}$ is a coarse moduli space). Again by definition of coarse moduli space, this morphism should factorize through $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d) \to \overline{M}_{0,n}(\mathbb{P}^r, d)$, so that we end up with the desired morphism $\overline{M}_{0,n}(\mathbb{P}^r, d) \to \overline{M}_{0,n}$.

Lemma 3.5. *The forget-the-map morphism is flat of relative constant dimension.*

The Lemma above implies that the pullback of boundary divisors $D(A|B)$ of $\overline{M}_{0,n}$ are divisors, and by construction they will be in the boundary of $\overline{M}_{0,n}(\mathbb{P}^r, d)$. How many irreducible components $F^{-1}D(A|B)$ has?

To answer this question, let ξ be a generic point of $F^{-1}D(A|B)$ (for those not familiar with the notion of generic point, simply think of a very small but dense open subscheme of $F^{-1}D(A|B)$). Over this there exists the generic curve $C|_\xi$ together with a map $\mu : C|_\xi \to \mathbb{P}^r$ of degree d.

The generic curve $C|_\xi$ is the curve over the generic point of $D(A|B)$, hence it has two irreducible components. Let d_i for $i = 1, 2$ be the degree of μ restricted to the i^{th} irreducible component. Then there exists an open subscheme in $F^{-1}D(A|B)$ where every point correspond to a map which has degree d_1 on one component and degree d_2 on the second one. It is immediate to check that the closure of this open subscheme cannot be the whole $F^{-1}D(A|B)$, as most of the morphisms where the degree of the map restricted to the first component is different from d_1 are not included.

Proposition 3.6. *The generic points of the irreducible components of $F^{-1}D(A|B)$ in $\overline{M}_{0,n}(\mathbb{P}^r, d)$ correspond to morphisms $\mu : C \to \mathbb{P}^r$ of degree d where C is the generic curve of $D(A|B)$ (hence it has two irreducible components) and the restriction of μ to the components has degree respectively d_1 and d_2, with $d_1 + d_2 = d$*

The other boundary components of $\overline{M}_{0,n}(\mathbb{P}^r, d)$ are those that surjects onto $\overline{M}_{0,n}$ via the forgetful morphism. The generic points of these divisors must correspond to morphisms $\mu : C \to \mathbb{P}^r$ where one of the two components of C is unstable, i.e. it has only one or zero markings.

For each of the two cases above, we deduce the existence of an irreducible component depending on the degree of the restrictions of the map. Be careful that the degree of the map on the unstable component must be > 0.

Proposition 3.7. *The irreducible components of $\partial\overline{M}_{0,n}(\mathbb{P}^r, d)$ are in bijection with the choices of partitions $A \cup B = \{1, \dots, n\}$ and $d_1 + d_2 = d$, with the additional condition that when $|A|$ or $|B|$ is ≤ 1, then respectively d_1 or d_2 must be > 0.*

As before, this discussion can be made more rigourous by properly exploiting the gluing morphisms

$$\overline{M}_{0,n+1}(\mathbb{P}^r, d) \times_{\mathbb{P}^r} \overline{M}_{0,m+1}(\mathbb{P}^r, e) \longrightarrow \overline{M}_{n+m}(\mathbb{P}^r, d + e)$$

As you might probably already had guessed, we will not be offering here such a detailed proof.

There is one last set of morphisms that we have to introduce before moving on to the next section, i.e. the *evaluation morphisms*. For $1 \leq i \leq n$ we have:

$$ev_i : \overline{M}_{0,n}(\mathbb{P}^r, d) \longrightarrow \mathbb{P}^r$$

which sends a stable map $(\mu : C \to \mathbb{P}^r)$ to the point $\mu(\sigma_i)$. The construction of this morphism can be obtained by constructing a natural transformation $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d) \to \mathbb{P}^r$ (as usual, the latter is identified with its functor of points), which is defined just as above.

But the coarse moduli space $\overline{M}_{0,n}(\mathbb{P}^r, d)$ has the property that every morphism from the stack to a scheme should factorize through it, hence we obtain the desired evaluation morphism.

Lemma 3.8. *The evaluation morhism* $ev_i : \overline{M}_{0,n}(\mathbb{P}^r, d) \to \mathbb{P}^r$ *is flat of relative constant dimension.*