Lecture 4: Kontsevich formula, Gromov-Witten invariants and quantum product

Disclaimer: these are *very* rough notes. Do not expect them to be exhaustive and/or detailed. There notes are intended more as a roadmap rather than a manual. Any comment is always welcome!

Guiding questions:

- How do we use the moduli space of stable maps to prove Kontsevich formula?
- What are Gromov-Witten invariants?
- What is a quantum product?
- How all these things do relate to each other?

4.1. **The Kontsevich formula.** In this section we sketch a proof for the Kontsevich formula. Recall:

Theorem 4.1. *The number N_d of degree d rational plane curves passing through $3d-1$ points in general position satisfies:*

$$N_d + \sum \binom{3d-4}{3d_A-1} d_A^2 N_{d_A} \cdot N_{d_B} \cdot d_A d_b = \sum \binom{3d-4}{3d_A-2} d_A N_{d_A} \cdot d_B N_{d_B} \cdot d_A d_B$$

where the sum is taken over the pairs (d_A, d_B) such that $d_A + d_B = d$ and both numbers are ≥ 1.

Consider the coarse moduli scheme $\overline{M} := \overline{M}_{0,3d}(\mathbb{P}^2, d)$. Fix $3d-2$ points $Q_1, \ldots, Q_{3d-2}$ in $\mathbb{P}^2$ and two distinct lines L_1 and L_2 such that the set of point formed by the Q_i together with $L_1 \cap L_2$ are in general position. Define:

$$Y := ev_1^{-1}(Q_1) \cap ev_2^{-1}(Q_2) \cap \cdots \cap ev_{3d-2}^{-1}(Q_{3d-2}) \cap ev_{3d-1}^{-1}(L_1) \cap ev_{3d}^{-1}(L_2)$$

Recall that evaluation morphisms are flat of relative constant dimension, hence the expected codimension of Y is $2 \cdot (3d-2) + 2 = 6d-2$. The next lemma tells us that this is the actual codimension of Y.

Lemma 4.2. *The scheme Y has the expected codimension, is wholly contained in $\overline{M}^*$ (the locus of maps with no non-trivial automorphisms) and intersects transversally the boundary components of $\overline{M}$.*

Let us comment on the last statement: the dimension of $\overline{M}$ is $3d-3+3(d+1)-1 = 6d-1$, from which we deduce that Y is a curve, so it makes perfect sense that we expect Y to intersect the boundary components. The Lemma assures us that it will do so transversally, hence in a finite number of points.

Consider the morphism $f : \overline{M} \to \overline{M}_{0,4}$ that forgets the map and only remembers $\sigma_1, \sigma_2, \sigma_{3d-1}$ and σ_{3d}. We have seen that this is a flat morphism too, and it implies that the (reducible) divisor $D(1,2|3d-1,3d) := f^{-1}D(\sigma_1,\sigma_2|\sigma_3,\sigma_4)$ is linearly equivalent to $D(1,3d-1|2,3d) := f^{-1}D(\sigma_1,\sigma_3|\sigma_2,\sigma_4)$. In particular we have:

$$|Y \cap D(1,2|3d-1,3d)| = |Y \cap D(1,3d-1|2,3d-4)| \tag{1}$$

By explicitly counting the intersection points on the two sides we will deduce the Kontsevich formula.

Remark 4.3. Ok, but *where* are you doing that intersection product, so to use the fact that linearly equivalent divisors intersect a curve in the same number of points? Here is a quick explanation for you: first, the variety $\overline{M}$ has only finite quotient singularities, hence it can be proved that it satisfies Poincaré duality or, equivalently, there is an intersection product in rational homology that it is dual to the cup product in cohomology (in general, there is a Poincaré duality for orbifolds).

If you prefer the language of stacks, the fact that $\overline{M}$ is the coarse moduli space for a smooth Deligne-Mumford stack implies that it inherits a product strucure on the rational Chow groups.

Whatever theory you choose, both of them satisfy the property that the intersection product in homology/Chow theory of two closed subschemes is equal to the schematic intersection whenever the latter is transversal and it is wholly supported in the smooth open locus.

We examine now all the possible intersection points of the left side of 1. From now on, the irreducible component A is the one containing $\sigma_{3d-1}, \sigma_{3d}$:

- Consider the locus where the map $\mu : C \to \mathbb{P}^r$ has degree 0 on the component containing σ_{3d-1} and σ_{3d}. Take an intersection point with Y: by construction $\mu(\sigma_{3d-1}) \in L_1$ and $\mu(\sigma_{3d}) \in L_2$, but the component get contracted! This implies that the component is contracted to $L_1 \cap L_2$, and none of the other markings can be on this component (otherwise one of the Q_i would coincide with $L_1 \cap L_2$).

  Therefore, $\mu(C)$ is a degree d rational curve passing through

  $$Q_1, \ldots, Q_{3d-2}, L_1 \cap L_2$$

  On the other hand, given such a curve, there is only one stable map $\mu : C \to \mathbb{P}^2$ up to isomorphism whose image is the given curve, because the contracted component has exactly three special points.

  Henceforth, the cardinality of the intersection of Y with this divisor is N_d.

- Consider the locus where the map $\mu : C \to \mathbb{P}^2$ has degree d_A on the A component. If the number of markings distinct from σ_{3d-1} and σ_{3d} on this component is $> 3d_A - 1$, we would have that the image $\mu(C_A)$ is a degree d_A-curve passing through more than $3d_A - 1$ fixed points, hence those points cannot be in general position, which contradicts our hypothesis on the set $\{Q_1, \ldots, Q_{3d-2}, L_1 \cap L_2\}$.

  If we have less than $3d_A - 1$ markings distinct from $\sigma_{3d-1}, \sigma_{3d}$ on C_A, the image curve $\mu(C_B)$ passes through more than $3d_B - 1$ of the points $\{Q_1, \ldots, Q_{3d-2}\}$, so again this contradicts the genericity assumption.

  When we have $3d_A - 1$ markings on C_A distinct from σ_{3d-1} and σ_{3d}, the image of $\mu(C_A)$ will be a degree d_A rational curve passing through $3d_A - 1$ points in $\{Q_3, \ldots, Q_{3d-2}\}$, and this is ok, and $\mu(C_B)$ will be a degree d_B rational curve passing through $3d_B - 1$ points of $\{Q_1, \ldots, Q_{3d-2}\}$.

  There are $\binom{3d-4}{3d_A-1}$ ways of choosing what markings among the spare ones should go on the A-component.

  Now we have to count how many stable maps $\mu : C \to \mathbb{P}^2$ there are such that $\mu(\sigma_i) = Q_i$ for $i \leq 3d-1$, $\mu(\sigma_{3d-1} \in L_1$ and $\mu(\sigma_{3d}) \in L_2$, assuming that the markings on each component are now decided.

  We have to decide what are the images of the markings σ_{3d-1}, σ_{3d} and of the node. The point $\mu(\sigma_{3d-1})$ has to be in $L_1 \cap \mu(C_A)$, which by the Bezout theorem has cardinality d_A, and $\mu(\sigma_{3d})$ has to be in $L_2 \cap \mu(C_A)$, which has the same cardinality. Therefore, we have d_A^2 possibilities. The node of C is sent by μ to any of the intersection points of $\mu(C_A)$ and $\mu(C_B)$: there are $d_A d_B$ of them.

  Now we are only left with counting how many curves passes through the fixed points plus the ones that we have selected as images of σ_{3d-1}, σ_{3d} and of the node: there are exactly $N_{d_A} N_{d_B}$ of them.

We have concluded the computation of the right side of 1, so now we move to the left side:

- Take an element in the intersection $Y \cap D(\sigma_1, \sigma_{3d-1} | \sigma_2, \sigma_{3d})$ such that $d_A = 0$, where d_A is the degree of the restriction of μ to C_A: then the C_A is contracted to a point, and the fact that it is contained in Y would imply that the point Q_1 belongs to the line L_1, which contradicts the general position assumption.

  We deduce that there are no points in $Y \cap D(\sigma_1, \sigma_{3d-1} | \sigma_2, \sigma_{3d})$ such that $d_A = 0$ or $d_B = 0$.

- Suppose $d_A > 0$ and let m_A be the number of markings on C_A. If $m_A > 3d_A$, then the image curve $\mu(C_A)$ is a degree d_A rational curve passing through more than $3d_A - 1$ points among the fixed ones, which contradicts the general position hypothesis.

  On the other hand, if $m_A < 3d_A$, a similar argument applied to the other component C_B gives the same conclusion. We deduce that the elements in $Y \cap D(\sigma_1, \sigma_{3d-1} | \sigma_2, \sigma_{3d})$ must have $3d_A$ markings on C_A and $3d_b$ markings on C_B.

- The number of ways of distributing $3d_A - 2$ markings among the $3d - 4$ markings (we are excluding here σ_1, σ_2, σ_{3d-1} and σ_{3d} which are already assigned to the components) is equal to $\binom{3d-4}{3d_A-2}$.

  The node can be sent to any of the intersection points of $\mu(C_A)$ with $\mu(C_B)$: there are $d_A d_B$ of them.

  The marking σ_{3d-1} can go to any of the points in $\mu(C_A) \cap L_1$, and similarly σ_{3d} can go to any of the points in $\mu(C_B) \cap L_2$: this gives $d_A d_B$ possibilities.

  Finally, once the points are choosen, there are exactly N_{d_A} possible images for $\mu(C_A)$ and N_{d_B} possible images for $\mu(C_B)$.

Putting all together, we deduce the Kontsevich formula.

4.2. **Gromov-Witten invariants of $\mathbb{P}^2$.** The moduli space of stable maps can be used to define *Gromov-Witten* invariants. Recall that in the proof of Kontsevich formula we used the evaluation morphisms

$$ev_i : \overline{M}_{0,n}(\mathbb{P}^2, d) \longrightarrow \mathbb{P}^2$$

to construct a curve Y inside $\overline{M}_{0,3d}(\mathbb{P}^2, d)$. The fact that Y was defined as the intersection of the preimages of some subvarieties (specifically, points and lines) of $\mathbb{P}^2$ allowed us to give a modular interpretation of this curve: namely, the stable maps contained in Y were the ones such that $\mu(\sigma_i) = Q_i$ for $i = 1, \ldots, 3d-2$ and $\mu(\sigma_{3d-1})$, $\mu(\sigma_{3d})$ lay respectively in the lines L_1 and L_2.

Then we intersected this curve with a divisor to produce, at the very end, a number.

The main idea behind Gromov-Witten invariants is to do the same thing, but this time producing a bunch of points inside $\overline{M}_{0,n}(\mathbb{P}^2, d)$ rather than curve, so that we can directly count them.

Several issues are on the way: even if we start with a set of subvarieties $\Gamma_1, \ldots, \Gamma_n$ whose codimensions sum up to $dim(\overline{M}_{0,n}(\mathbb{P}^2, d)) = n + 3d - 1$, and even if we know that $ev_i^{-1}(\Gamma_i)$ will have the same codimension because of flatness of ev_i, there is no reason why the intersection

$$ev_1^{-1}(\Gamma_1) \cap \cdots \cap ev_n^{-1}(\Gamma_n)$$

should has the expected codimension (i.e. should be transversal).

Remark 4.4. We haven't mentioned so far the existence of moduli of stable maps from higher genus curves to a variety X, but in these other cases way more problems manifest themselves: namely, the moduli space is no more irreducible, is not purely dimensional hence is not even clear what is the correct codimension to obtain a bunch of point and in case in what components we should count the intersection points.

These type of issues had been solved by Behrend and Fantechi via the definition of the so-called *virtual fundamental class*.

The solution to this problem is to move from intersecting subvarieties to intersect homology classes (in the sense of intersection product), and instead of counting the points we compute the degree of the resulting dimension zero cycle.

But here comes another problem: no intersection product in homology/chow groups for singular varieties, in general. Luckily, for orbifolds we do have an intersection product for *rational* chow/homology groups.

Remark 4.5. As already mentioned before, the existence of an intersection product for rational chow/homology groups is a consequence of the following fact: the moduli *stack* $\mathcal{M}_{0,n}(\mathbb{P}^2, d)$ of stable maps with n markings and of genus 0 is a smooth Deligne-Mumford (quotient) stack, hence it admits a well defined intersection product for chow/homology groups.

Moreover, such groups, taken with rational coefficients, are isomorphic to the ones of its coarse moduli space, that is $\overline{M}_{0,n}(\mathbb{P}^2, d)$, hence the latter inherits an intersection product.

Another possible approach to the problem of defining intersection product on singular varieties is given by *operational chow/homology groups*.

In this way we are actually able to produce a number out of n subvarieties of $\mathbb{P}^2$ whose codimensions sum up to $dim(\overline{M}_{0,n}(\mathbb{P}^2, d))$. We set:

$$I_d([\Gamma_1], \ldots, [\Gamma_n]) := \int_{\overline{M}_{0,n}(\mathbb{P}^2,d)} ev_1^*[\Gamma_1] \cdot \cdots \cdot ev_n^*[\Gamma_n]$$

Here the integral symbol must be understood in the following sense: look at the proper morphism $p : \overline{M}_{0,n}(\mathbb{P}^2, d) \to pt$ and take the pushforward along p: the result will be a (rational) multiple of the class of the point, and we take that number as value of the integral.

Definition 4.6. Let $\gamma_1, \ldots, \gamma_n$ be elements in $A^*(\mathbb{P}^2)$ (the latter can be understood either as rational Chow ring or rational homology ring). Then the *associated degree d Gromov-Witten invariant with n markings* is:

$$I_d(\gamma_1, \ldots, \gamma_n) := \int_{\overline{M}_{0,n}(\mathbb{P}^2,d)} \gamma_1 \cdot \cdots \cdot \gamma_n$$

Observe that the definition above is well posed even if the sum of the codimensions/degrees of γ_i is not equal to $dim(\overline{M}_{0,n}(\mathbb{P}^2, d))$. In that case the integral is simply equal to zero. Even more, the elements γ_i don't have to be homogeneous for the definition above to make sense.

Remark 4.7. When $\gamma_i = [\Gamma_i]$ are cycle classes of actual subvarieties which are in general positions, we can interpret the Gromov-Witten invariants of degree d as the number of degree d rational plane curves incident to the subvarieties Γ_i.

Here is a list of some properties of GW invariants that are rather easy to prove:

(1) GW invariants are $\mathbb{Q}$-linear and symmetric with respect to the entries $\gamma_1, \ldots, \gamma_n$.
(2) The non-zero GW invariants of degree 0 must have three markings.
(3) All GW invariants with more than 3 markings and where one of the entries is $1 = [\mathbb{P}^2]$ are zero: to prove this, use the fact that, for $n > 2$, the pushforward of 1 along the forget-the-marking morphism

$$\overline{M}_{0,n+1}(\mathbb{P}^2, d) \longrightarrow \overline{M}_{0,n}(\mathbb{P}^2, d)$$

is zero.
(4) $I_d(\gamma_1, \gamma_2, \gamma_3) = \int_{\mathbb{P}^2} \gamma_1 \cdot \gamma_2 \cdot \gamma_3$.
(5) $I_d(\gamma_1, \ldots, \gamma_n, h) = I_d(\gamma_1, \ldots, \gamma_n)d$: to prove this, use the fact that the pushforward of $[ev_{n+1}^{-1}(H)]$ along the forget-the-marking morphism is equal to $d[\overline{M}_{0,n}(\mathbb{P}^2, d)]$.

Remark 4.8. Observe that $I_d(h^2, \ldots, h^2) = N_d$, where the number of entries is $3d - 1$.

4.3. **Quantum product.** GW invariants can be used to construct the so called *quantum product* in the *quantum cohomology ring* $A^*(\mathbb{P}^2) \otimes \mathbb{Q}[[x_0, x_1, x_2]]$. A quantum object is usually an enhancement of the original object that depends on some paramenters x_i (or q_i) which specialize to the original object when we set $x_i = 0$ for every i.

As we already mentioned, the quantum cohomology of $\mathbb{P}^2$ is simply $A^*(\mathbb{P}^2) \otimes \mathbb{Q}[[x_0, x_1, x_2]]$. The quantum product of two elements in the quantum cohomology ring is completely determined by the quantum product $\gamma_1 * \gamma_2$ of two elements in $A^*(\mathbb{P}^2)$. We would like this quantum product to satisfy the following conditions:

(1) $1 = [\mathbb{P}^2]$ is the identity for $*$.
(2) Commutativity.
(3) Associativity.
(4) For every element α, β we have $\alpha|_{\underline{x}=0} * \beta|_{\underline{x}=0} = \alpha \cdot \beta$, where the latter is the usual intersection product.

Surprisingly, associativity turns out to be not that easy to prove. Set

$$I(\gamma_1 \cdots \gamma_n) = \sum_{d \geq 0} I_d(\gamma_1 \cdots \gamma_n)$$

Let us introduce the following notation:

- For any $\underline{a} = (a_0, a_1, a_2)$ 3-tuple of natural numbers, set
$$\underline{h}^{\underline{a}} := (h^{a_0}, h^{a_1}, h^{a_2})$$
- Similarly, set $\underline{x} = (x_0, x_1, x_2)$ and $\underline{x}^{\underline{a}} = (x_0 a_0, x_1^{a_1}, x_2^{a_2})$.
- Finally, define $\underline{a}! = a_0! a_1! a_2!$.

Then we define the *Gromov-Witten potential* as:

$$\Phi := \sum_{\underline{a}} \frac{\underline{x}^{\underline{a}}}{\underline{a}!} I(\underline{h}^{\underline{a}})$$

This is actually a finite sum, as can be easily checked using the properties of GW invariants. Observe that we can take partial derivatives of this potential with respect to x_i, by simply applying the derivation rules for formal power series. In particular:

$$\Phi_{ijk} = \sum_{\underline{a}} \frac{\underline{x}^{\underline{a}}}{\underline{a}!} I(\underline{h}^{\underline{a}}, h^i, h^j, h^k)$$

Then the quantum product of two elements is determined by the following formula:

$$h^i * h^j := \sum_{e+f=2} \Phi_{ije} \cdot h^f$$

We can decompose the potential Φ as a sum of its *classical* part and its *quantum* part. Namely, the classical part Φ^{cl} corresponds to the cubic polynomial

$$\Phi^{cl} = \sum_{\underline{a}} \frac{\underline{x}^{\underline{a}}}{\underline{a}!} I_0(\underline{h}^{\underline{a}})$$

If $I_+(-)$ denotes the sum of $I_d(-)$ for $d > 0$, then the quantum part is defined as:

$$\Gamma = \sum_{\underline{a}} \frac{\underline{x}^{\underline{a}}}{\underline{a}!} I_+(\underline{h}^{\underline{a}})$$

We can thus decompose the quantum product as follows:

$$h^i * h^j = \sum_{e+f=2} \Phi^{cl}_{ije} h^f + \sum_{e+f=2} \Gamma_{ije} h^f$$

A straightforward computation implies that the classical part of the quantum product its just $h^i \cdot h^j$.

Let us show how associativity of the quantum product implies the Kontsevich formula. We have the equality:

$$h * (h * h^2) = (h * h) * h^2$$

After unpacking the definition of quantum product, we get:

$$\Gamma_{121}(h^2 + \Gamma_{111}h + \Gamma_{112}h^0) + \Gamma_{122}h^1 = \Gamma_{221}h + \Gamma_{222}h^0 + \Gamma_{111}(\Gamma_{121}h + \Gamma_{122}h^0) + \Gamma_{112}h^2$$

If we restrict to the coefficients of h^0 we deduce:

$$\Gamma_{112}\Gamma_{121} = \Gamma_{222} + \Gamma_{111}\Gamma_{122}$$

Observe that $\Gamma_{222} = N_d$. Actually, after unpacking the various elements of the equations above, we retrieve the Kontsevich formula.