Lecture 5: Topological quantum field theories and GW-invariants

Disclaimer: these are *very* rough notes. Do not expect them to be exhaustive and/or detailed. There notes are intended more as a roadmap rather than a manual. Any comment is always welcome!

Guiding question:

- What is the connection between GW-invariants, quantum product and physics?

5.1. **Topological quantum field theories.** We survey here the concept of topological quantum field theory (TQFT), how it has arisen from physics and how it can be mathematically formalized.

5.1.1. *Background from physics.* In classical mechanics, the evolution of a system (say, a free particle in some space) is modeled as a path on a variety that represents space-time. The principle of least action in particular says the following: given some initial conditions (e.g. initial position and speed of the particle), the path representing the dynamic of the system from the initial time to a fixed time will be the one that minimizes the value of a certain functional. The latter functional, usually called the action, depends on what kind of forces play a role in the system.

We can consider the variety representing space-time as an element of a specific "geometrical" category: usually symplectic, differential or topological. For instance, in a general relativistic setting, the "action" of gravity on the evolution of the system is encoded in the curvature of the space-time, that is in the Riemannian metric of the variety: in this case, we hence look at the variety as a Riemannian manifold. The Hamiltonian of the system is usually encoded into a symplectic form, thus in this case we move into the symplectic realm. When the system is invariant with respect of the choice of a symplectic or a Riemannian structure, usually what remains is only the topology of the space-time. We will get back to this in a moment.

In quantum mechanics, we shift from dynamics on the space-time to dynamics of the wave functions on the space-time. Adopting the Schr'odinger approach, the wave function evaluated at a certain point in the space-time, gives you the probability that the corresponding event occurs (e.g. the particle being at a specified position and having a fixed momentum at a certain time). Feynman and others proposed a *variational* way, akin to the principle of stationary action, to construct the wave function, that is to compute the probability of an event: if S denotes the action functional, you have to "integrate" the values of e^{iS} along all the possible paths which join your initial state and the final state. The squared modulus of the resulting complex number (which is called probability amplitude) will give you the probability that your system evolves into the choosen final state.

Another relevant role is played by entaglement phenomena: start with an ambient space, and then divide it into two non-communicating subspaces by inserting some barrier. Then if you begin with a wave function ϕ defined on your ambient space, after inserting the barrier you will end up with a wave function which is a sum of tensor product of wave functions on the subspaces, i.e. something like $\sum_{i,j} < \phi_i(x), \phi_j(y) > \phi_i(x) \otimes \phi_j(y)$ (the states of the two systems, even if non-communicating, are now *entangled*).

The coefficients $< \phi_i, \phi_j >$ appearing in the sum above can be explained in terms of *correlation* of the states ϕ_i and ϕ_j: they measures what is the probability for the two distinct elements of the system of being the first in one state and the second in another one. These correlation functions can also be computed using the path-integral approach. One can also generalize this concept to n-correlation functions, which are usually denoted $< \phi_1(x_1), \cdots, \phi_n(x_n) >$.

Finally, we also have a distinguished state or wave function, the so called *vacuum state*: is the state of the system when there is no energy.

The upshot of the preceeding discussion is that a quantum system is encoded by the following pieces of data:

- A vector space $Z(\Sigma)$ of wave functions or states for every physical space Σ.
- A way to compute, given a certain evolution of Σ in the time and an initial state ϕ_0, what is the final state ϕ_f of the system. Such evolution may be represented by a cylinder $\Sigma \times I$ but also by some more complicated $n+1$-dimensional oriented manifold, as it happens for instance when we divide the ambient space into other spaces by means of solid barriers (or when we remove them). In other terms, we have a morphism $Z(M) : Z(\Sigma_0) \to Z(\Sigma_f)$.
- A vacuum state $\not\Vdash$ in $Z(\Sigma)$, corresponding to the evoution from $\emptyset$ to Σ. In particular $Z(\emptyset) = \mathbb{C}$ or whatever ground field.

The correlation functions are somehow encoded in the second and the third piece of data: if the border of M is given by $\Sigma \sqcup \Sigma$ on the ingoing side and $\emptyset$ on the outgoing side (w.r.t. the time flow), the assumptions of quantum mechanics tells us that $Z(\Sigma \sqcup \Sigma) = Z(\Sigma) \otimes Z(\Sigma)$, and the manifold above gives us an element of $(Z(\Sigma)^\vee)^{\otimes 2}$, if we interpret the states on Σ with ingoing boundary as the dual of the states on Σ with outgoing boundary. This element is exactly the 2-correlation function.

5.1.2. *Axiomatization of TFQT.* The preceeding discussion should make the following axioms of TQFT pretty natural. The formulation below is due to Atiyah, who was in turn inspired by the approach of Segal to Conformal Quantum Field Theory.

Choose a positive orientation for every n-dimensional, closed, orientable manifold with no boundary. Then the category of n-dimensional cobordisms is the category whose objects are oriented, closed n-manifolds Σ with no boundary, and whose morphisms $\Sigma_1 \to \Sigma_2$ are given by $n+1$-dimensional, oriented and closed manifold M such that $\partial M = \Sigma_1^- \sqcup \Sigma_2$, where Σ_1^- denotes the manifold Σ_1 negatively oriented.

Definition 5.1. A *Topological Quantum Field Theory* (TQFT) is a functor $Z : (\mathrm{Cobord})_n \to (\mathrm{Vect})_k$ satisying the following axioms:

(1) Z is functorial with respect to diffeomorphism of n-manifolds and $n+1$-manifolds preserving the orientation.
(2) $Z(\Sigma^-) = Z(\Sigma)^\vee$.
(3) $Z(\Sigma_1 \sqcup \Sigma_2) = Z(\Sigma_1) \otimes Z(\Sigma_2)$.
(4) If M is an $n+1$-manifold with $\partial M = \Sigma_1^- \sqcup \Sigma_3$ is obtained by gluing two manifolds M_1 with $\partial M_1 = \Sigma_1^- \sqcup \Sigma_2$ and M_2 with $\partial M_2 = \Sigma_2^- \sqcup \Sigma_3$ at Σ_2, then $Z(M_2) \circ Z(M_1) = Z(M)$.
(5) $Z(\emptyset_n) = k$.
(6) $Z(\emptyset_{n+1}) = \mathrm{id}$.
(7) $Z(\Sigma \times I) = \mathrm{id}$.

It is an easy consequence of the axioms above that Z must also be homotopy invariant. Observe that Z defines for every $n+1$-dimensional, oriented, closed manifold M with no boundary a topological invariant $Z(M)$.

5.1.3. *TQFT and Frobenius algebras.* From now on we will focus on the case $n = 1$.

Let Z be a TQFT and set $A := Z(S^1)$. This vector space has a multiplication $\bullet : A \otimes A \to A$ induced by the 2-manifold usually called *pair of pants*, i.e. a sphere with three punctures, two of them with incoming orientation and the other with outgoing orientation. Playing around with the axioms of TQFT one can prove that

$(A, \bullet, 1)$ has the structure of k-algebra, where the element 1 is determined by the sphere having one puncture with outgoing orientation.

We also have a trace morphism $tr : A \to k$ given by the sphere having one puncture with incoming orientation, and a pairing $<,>: A \otimes A \to k$ defined by the two-punctured sphere, where the punctures have incoming orientation. Playing around with the axioms of TQFT one can prove:

Proposition 5.2. *The k-algebra $(A, \bullet, 1)$ is a* Frobenius algebra*, i.e. a commutative, associative k-algebra with unity endowed with a trace form $tr : A \to k$ that defines a non-degenerate symmetric pairing $< a, b >:= tr(a \bullet b)$.*

Observe that the pairing induces an isomorphism $s : A \simeq A^\vee$.

Consider the surfaces of genus 1 with one incoming puncture. This defines a morphism $< - >_1: A \to k$ (we can interpret it as the "expectation value" that a state evolves into the vacuum state after inserting and removing a barrier). This surface can be constructed by gluing two pair of pants with opposite orientation and then closing the outgoing boundary with a disk. The axioms of TQFT then implies that:

$$< \phi >_1 = < (\sigma^{-1} \circ \bullet \circ \sigma)(\phi) >$$

In other terms, we can reconstruct this quantity from the structure of Frobenius algebra A. Actually, more is true:

Theorem 5.3 (Abrams)**.** *There is an equivalence of categories*

$$(\mathrm{1D} - \mathrm{TQFT}) \simeq (\mathrm{FrobAlg})$$

5.2. **TQFT and quantum cohomology.**

5.2.1. *Small quantum cohomology.* Let X be a smooth projective variety over $\mathbb{C}$ with vanishing odd cohomology (this assumption can be dropped but the theory below need to be slightly modified) and define the *Novikov ring* Λ as the subring

$$\Lambda \subset \mathbb{C}\{\{e^\delta\}\}, \quad \delta \in H_2(X)$$

of formal power series $\sum_\delta a_\delta e^\delta$ satifying the following property: for every real number C, among all the δ such that $\deg(\delta) \leq C$ only finitely many coefficients a_δ are non-zero.

Here $H_2(X) := H_2(X, \mathbb{Z})/\mathrm{tors}$ and $H^\bullet(X)$ denotes the singular cohomology with $\mathbb{C}$-coefficients of X. We use the notation e^δ so to suggest the multiplication rule $e^\delta \cdot e^\gamma = e^{\delta+\gamma}$ and $e^{n\delta} = (e^\delta)^n$ for any integer n.

We define the *small quantum cohomology ring* $QH^\bullet(X)$ as the vector space $H^\bullet(X) \otimes \Lambda$ endowed with the *small quantum product*

$$\alpha * \beta := \sum_{\delta \in H_2(X)} e^\delta \left(\sum_{\gamma \in H^\bullet(X)} I_\delta(\alpha, \beta, \gamma)\gamma^\vee \right)$$

where:

(1) $I_\delta(\alpha, \beta, \gamma) := \int_{\overline{M}_{0,3}(X,\delta)} ev_1^*\alpha \cdot ev_2^*\beta \cdot ev_3^*\gamma$ is the analogue of the GW-invariants that we discussed in Lecture 4, with the difference that the target variety here is no more $\mathbb{P}^2$ but a general smooth variety X, hence we cannot use the degree to specify the cycle class of the image of the stable maps but directly the cycle class δ in $H_2(X)$.
(2) $\gamma^\vee$ is the dual of γ in $H^\bullet(X)$ with respect to the intersection pairing form.

Observe that the small quantum product and the small quantum cohomology ring differs from the quantum product and quantum cohomology introduced in Lecture 4, although the key ingredients for its definition are again GW-invariants.

The small quantum product extends in an obvious way to the whole $QH^\bullet(X)$. Moreover, we can also extend the intersection pairing on $H^\bullet(X)$ to $QH^\bullet(X)$ by setting $< \alpha e^\delta, \beta e^\gamma >:= 0$ for $\delta \neq 0$ or $\gamma \neq 0$.

Proposition 5.4. *The small quantum cohomology ring $QH^\bullet(X)$ together with the quantum product $*$ and the intersection pairing $<,>$ is a Frobenius algebra.*

The Proposition above combined with Abrams classification result shows that the small quantum cohomology ring can be used to construct a 1D-TQFT. In particular we have $Z(S^1) = QH^\bullet(X)$ and the pairings basically depends only on the Gromov-Witten invariants of X.