Topology of algebraic varieties - 1st lecture

Outline:

- Cohomology of algebraic varieties
  - Hodge theory
  - Lefschetz theorems
  - Correspondences

- Algebraic cycles
  - algebraic correspondences
  - Conjectures

Cohomology of algebraic varieties

X smooth proj variety of dimension n $\longmapsto$ $H^k(X, \mathbb{Q})$ singular cohom. w/ $\mathbb{Q}$-coefficients

$$H^*(X,\mathbb{Q}) = \bigoplus_{k=0}^{2n} H^k(X, \mathbb{Q})$$

$\downarrow$

$H^k(X, \mathbb{C}) \cong H^k(X, \mathbb{Q}) \otimes \mathbb{C}$
singular coh. w/ complex coeff

$\Omega^{p,q}(X) =$ (p,q)-differential forms $\quad \bar{\partial}_{p,q} : \Omega^{p,q} \to \Omega^{p,q+1}$

$H^{p,q}(X) := \ker(\bar{\partial}_{p,q}) / \operatorname{im}(\bar{\partial}_{p,q-1})$ (Dolbeault cohomology)

Thm (Hodge)

$$H^k(X,\mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}(X)$$

$$\overline{H^{p,q}(X)} \simeq H^{q,p}(X) \quad \text{(Hodge duality)}$$

symmetry wrt red axis (Serre duality)

symmetry wrt blue axis (Hodge duality)

$$
\begin{array}{ccc}
H^{0,0}(X) & \cong & H^0(X,\mathbb{C}) \\
H^{1,0}(X) \oplus H^{0,1}(X) & \cong & H^1(X,\mathbb{C}) \\
H^{2,0} \oplus H^{1,1}(X) \oplus H^{0,2} & \cong & H^2(X,\mathbb{C}) \\
\cdots & & \vdots \\
H^{n,0}(X) \oplus H^{n-1,1}(X) \cdots H^{1,n-1}(X) \oplus H^{0,n}(X) & \cong & H^n(X,\mathbb{C}) \\
\cdots & & \vdots \\
H^{n,n}(X) & \cong & H^{2n}(X,\mathbb{C})
\end{array}
$$

Lefschetz theorem

X sm proj var $\Rightarrow$ $\mathcal{L} \to X$ very ample line bdl

$$\begin{array}{c} L \in H^{1,1}(X,\mathbb{C}) \cap H^2(X,\mathbb{Q}) \\ \shortparallel \\ c_1(\mathcal{L}) \end{array}$$

hyperplane section

(if $X \hookrightarrow \mathbb{P}^N = \mathbb{P}H^0(X,L)$ then $L = H \cap X$ where H is a (suitable) hyperplane in $\mathbb{P}^N$)

Via the cup product we can define:

$$L : H^k(X) \to H^{k+2}(X)$$
$$\xi \mapsto L \cup \xi$$

Thm (Lefschetz) $(k \leq n = \dim X)$

$$L^{n-k} : H^k(X) \xrightarrow[\cong]{} H^{2n-k}(X) \quad \text{isomorphism}$$

$$\Downarrow$$

$$\cdots \to H^{k-2}(X) \underset{L}{\hookrightarrow} H^k(X) \underset{L}{\hookrightarrow} H^{k+2}(X) \underset{L}{\hookrightarrow} \cdots \qquad \cdots \underset{L}{\twoheadrightarrow} H^{2n-k-2}(X) \underset{L}{\twoheadrightarrow} H^{2n-k}(X) \underset{L}{\twoheadrightarrow} H^{2n-k+2}(X) \to \cdots$$

Def $H^k(X)_{prim} := \ker\left(H^k(X) \xrightarrow[L^{n-k+1}]{} H^{2n-k+2}(X) \right)$

primitive cohomology

Thm (Lefschetz decomposition)

$$H^k(X) \cong \bigoplus_{r=0}^{\lfloor k/2 \rfloor} L^r H^{k-2r}(X)_{prim}$$

Hodge index thm

X smooth, hence $H^*(X)$ w/ cup product is a ring

We can define:

$$(\, , \,)_L : H^k(X) \otimes H^k(X) \longrightarrow \mathbb{Q}, \quad \alpha \otimes \beta \longmapsto \int_X \alpha \cup L^{n-k} \beta$$

We can use $\langle\,,\rangle_L$ to define a hermitian form:

$$H(-,-) : H^k(X,\mathbb{C}) \otimes H^k(X,\mathbb{C}) \longrightarrow \mathbb{C}$$
$$\alpha \otimes \beta \longmapsto i^k \int_X \alpha \cup L^{n-k} \bar{\beta} = i^k \langle \alpha, \bar{\beta} \rangle_L$$

Thm (Hodge index thm)

$H(\cdot,\cdot)$, restricted to $H^{p,q}(X) \cap H^k(X)_{prim}$ is definite of sign $(-1)^p$

Correspondences

Thm (Künneth)

$$H^k(X \times Y) \cong \bigoplus_{p+q=k} H^p(X) \otimes H^q(Y)$$

$$\Downarrow$$

Poincaré/Serre duality

$$H^k(X \times Y) \cong \bigoplus_{p+q=k} H^p(X) \otimes H^q(Y) \cong \bigoplus_{p+q=k} H^{2n-p}(X)^\vee \otimes H^q(Y)$$

$$\cong \bigoplus_{p+q=k} \mathrm{Hom}\left(H^{2n-p}(X), H^q(Y)\right)$$

Q: how does this is works concretely?

Consider $\xi \in H^p(X) \otimes H^q(Y) \subseteq H^k(X \times Y)$. Can define:

$$\boxed{\xi_* : H^{2n-p}(X) \to H^q(Y), \quad \alpha \longmapsto pr_{2*}(\xi \cup pr_1^* \alpha)}$$

This induces $H^p(X) \otimes H^q(Y) \cong \mathrm{Hom}(H^{2n-p}(X), H^q(Y))$

Moreover $H^p(X) \otimes H^q(Y) \cong H^p(X) \otimes H^{2m-q}(Y)^\vee$

$$\cong \mathrm{Hom}(H^{2m-q}(Y), H^p(X))$$

This iso is induced by: $\boxed{\xi^*(\beta) := pr_{1*}(\beta \cup pr_2^* \xi)}$

Def

Given $\xi \in H^k(X \times Y)$, the induced morphism ξ_* is **the induced correspondence** (also ξ^* is a correspondence, because $\xi^* = (\iota^* \xi)_*$, where

$$\iota : Y \times X \longrightarrow X \times Y$$
$$(y, x) \longmapsto (x, y) \quad)$$

By Künneth thm, every $\varphi : H^p(X) \to H^q(Y)$ is actually induced by $\xi \in H^{2n-p+q}(X \times Y)$, ie

$$\varphi = \xi_*$$

E.g.

$\Delta \in H_n(X \times X) \cong H^n(X \times X)$

$\Delta \subseteq X \times X$ diagonal $\Rightarrow$ $\Delta_* = \mathrm{id} : H^k(X) \to H^k(X)$

$f : X \to Y$, $\Gamma_f \subseteq X \times Y$ graph

$\Rightarrow$ $\Gamma_{f*} = f_* : H^k(X) \to H^{2m-2n+k}(Y)$

$\Gamma_f^* = f^* : H^k(Y) \to H^k(X)$

$\Gamma_f \in H_n(X \times Y)$
$\cong H^{2m}(X \times Y)$

$m = \dim(Y)$

2nd part: algebraic cycles and conjectures (your way to become rich and famous)

Algebraic cycles

X smooth proj variety

$Z \subseteq X$ subvariety $\rightsquigarrow$ you can think of Z also as a cohomology class

$Z \subseteq X$ of dimension $n-k$

$\downarrow$

$[Z] \in H_{2n-2k}(X) \underset{\text{Poincare duality}}{\rightsquigarrow} [Z] \in H^{2k}(X)$

We are defining: $\underset{\ni a_i \in \mathbb{Q}}{\sum a_i Z_i} \longmapsto \sum a_i [Z_i]$ cohomology classes

$$cl : \mathcal{Z}^k(X)_{\mathbb{Q}} \longrightarrow H^{2k}(X, \mathbb{Q})$$

abelian grp generated by $Z \subseteq X$ subvars of codim k

image of cl are the algebraic coh classes

Question

Given $\xi \in H^{2k}(X,\mathbb{Q})$, can we say if ξ is algebraic?

i.e. $\xi \in \operatorname{im}(cl : Z^k(X) \to H^{2k}(X))$

i.e. $\xi \in H^{2k}(X,\mathbb{Q})_{alg}$?

A necessary condition

$$\xi \in H^{2k}(X,\mathbb{Q})_{alg} \Rightarrow \xi \in H^{2k}(X,\mathbb{Q}) \cap H^{k,k}(X,\mathbb{C})$$

Is this condition also sufficient?

if $Z \in H^{2k}(X \times X)$ is algebraic then $[Z]_*$ sends (p,q)-forms to $(p+k-n, q+k-n)$-forms

Hodge conj for correspondences: does the viceversa hold?

(Some) standard conjectures

Künneth conjecture : $[\Delta]_* = id$ $(\Delta \subseteq X \times X$ diagonal$)$

$$id = \sum \delta_k \quad \text{where } \delta_k = \begin{cases} id & \text{on } H^k(X) \\ 0 & \text{otherwise} \end{cases}$$

Therefore: $\delta_k = [D_k]_*$, $D_k \in H^{2n}(X \times X)$

$\Rightarrow \Delta = \sum D_k$ in $H^{2n}(X \times X)$

Conj: D_k are algebraic

Lefschetz conjecture: $L^{n-k} : H^k(X) \xrightarrow{\simeq} H^{2n-k}(X)$

$\Rightarrow \Lambda_{n-k} : H^{2n-k}(X) \xrightarrow{\simeq} H^k(X)$

$\Lambda_{n-k} := (L^{n-k})^{-1}$ $\Rightarrow \exists\, \lambda_{n-k} \in H^{2k}(X \times X)$

$[\lambda]_* = \Lambda_{n-k}$

Conj: λ_{n-k} are algebraic.

Hom = Num conjecture: Recall that we have a pairing

$$H^k(X) \otimes H^{2n-k}(X) \xrightarrow{\langle , \rangle} \mathbb{Q}$$

$$\alpha \otimes \beta \longmapsto \int \alpha \cup \beta$$

We can say that

$$\alpha \in H^k(X) \text{ is } \underset{\text{numerically equivalent}}{\equiv} 0 \quad \text{iff} \quad \langle \alpha, - \rangle \equiv 0$$

If $\alpha \equiv 0$ then α is also homologically equivalent to 0 ($\alpha = 0$ in $H^{k}(X, \mathbb{Q})$).

Conj: if α is algebraic, then

$$\alpha \equiv 0 \iff \alpha = 0$$

numerically equivalent

homologically equivalent