Recap

$\xi \in H^*(X \times Y) \Rightarrow \xi_* : H(X) \longrightarrow H(Y)$

$\alpha \longmapsto pr_{2*}(pr_1^* \alpha \cup \xi)$

$Z = \sum a_i Z_i \quad a_i \in \mathbb{Q}$ $\quad$ ($\subset X \times Y$) $\qquad \xi^* : H(Y) \longrightarrow H(X)$

$\alpha \longmapsto pr_{1*}(pr_2^* \alpha \cup \xi)$

cycle $\Rightarrow [Z] \in H(X \times Y)$

$\Rightarrow [Z]_*, [Z]^*$ $\longleftarrow$ algebraic correspond.

(Some) Standard conjectures

Lefschetz: $\Lambda_{n-k} : H^{2n-k}(X) \xrightarrow{\sim} H^k(X)$

(inverse of $L^{n-k} : H^k(X) \to H^{2n-k}(X)$)

L hyperplane section

Conj: Λ_{n-k} is algebraic

(i.e. $\exists \lambda_{n-k} \in H^{2k}(X \times X)_{alg}$ st
$[\lambda_{n-k}]_* = \Lambda_{n-k}$)

Homological vs Numerical

$[Z]$ algebraic coh class in $H^k(X)$

$$\langle [Z], - \rangle : H^{2n-k}(X) \longrightarrow \mathbb{Q}$$
$$\alpha \longmapsto \int_X [Z] \cup \alpha$$

<u>Def</u>

An alg coh class $[Z]$ is $\overset{\text{numerically}}{\equiv} 0$ if $\langle [Z], - \rangle \big|_{H^{2n-k}_{alg}}$ is zero.

<u>Conj</u>: $[Z] \equiv 0 \Rightarrow [Z] = 0$

Künneth conjecture

$$\delta_k = \begin{cases} id & \text{on } H^k(X) \\ 0 & \text{on } H^j(X),\ j \neq k \end{cases}$$

$\Delta_X = \sum \delta_i$ in $H^{2n}(X \times X)$

Conjecture: δ_k is algebraic

(actually known for δ_0 and δ_{2n})

Voisin Conjecture

Let X be sm proj var of dim n.

Let $Y \subseteq X$ be an algeb. subvariety

Let Z be an algebraic cycle st

this implies that $[Z] = i_* \mathfrak{z}$, $\mathfrak{z} \in H(Y)$

$$[Z]\big|_{H(X \setminus Y)} = 0 \qquad i : Y \hookrightarrow X$$

Then $\exists$ Z' algebraic cycle of Y such that $i_*[Z'] = [Z]$.

Lemma

The Lefschetz conjecture does not depend on the polarization L.

Actually, if $\exists Z$ alg cycle on $X \times X$ st

$$[Z]_* : H^{2n-k}(X) \longrightarrow H^k(X)$$

is an isomorphism $\Rightarrow$ Lefschetz conjecture is true.

Proof

by Cayley Ham

$$[Z]_* \circ \left(L^{n-k} \circ [Z]_* \right)^{-1} : H^{2n-k}(X) \longrightarrow H^k(X)$$

is an inverse of L^{n-k}, but $\left(L^{n-k} \circ [Z_*] \right)^{-1}$ is alg

$A : V \to V$ automorphism $\Rightarrow$ A^{-1} is a polynomial in A

Indeed, let $p_A(t)$ be the characteristic polynomial.

$$\Rightarrow \text{Cayley thm: } \underbrace{p_A(A)}_{\substack{\parallel \\ 0}} = \sum_{i=0}^{n} c_i A^i, \quad c_0 \neq 0$$

$$\Rightarrow \quad \mathrm{id} = \underbrace{\left(\frac{\sum_{i=1}^{n} c_i A^{i-1}}{-c_0} \right)}_{\star} \cdot A \Rightarrow A^{-1} = \star$$

Lefschetz $\Rightarrow$ Hom = Num

$\xi \in H^k(X) \Rightarrow$ by Lefschetz decomposition theorem

$$\xi = \sum L^i \xi_i \ , \quad \xi_i \in H^{k-2i}(X)_{prim}$$

We can define:

$$s : H^k(X) \to H^k(X), \quad s(\xi) = \sum (-1)^i L^i \xi_i .$$

We can use s to prove Hom = Num

We can define the following pairing on $H^\cdot(X)$

$$H^k(X) \otimes H^k(X) \to \mathbb{C}$$

$$\alpha \otimes \beta \longmapsto i^k \int_X \alpha \cup L^{n-k} \, s \overline{\beta}$$

By Hodge index thm, this "twisted" pairing is positive definite.

lemma: Lefschetz $\Rightarrow$ s algebraic.

If s is algebraic, the restricted pairing

$$H^k(X)_{alg} \otimes H^k(X)_{alg} \to \mathbb{C}$$

is non-singular.

Suppose $\exists Z$ st $\langle [Z], - \rangle = 0$ $([Z] \equiv 0)$

$$\Rightarrow \quad 0 = i^k \int_X Z \cup \underbrace{L^{n-k} s \bar{Z}} \Rightarrow [Z] = 0$$

because this is algebraic

because the pairing is >0.

Lefschetz $\Rightarrow$ Voisin

smooth alg subvar

Z alg cycle on X, $Z = i_* \xi$ where $i : Y \hookrightarrow X$, $\xi \in H(Y)$

Let $\tilde{Q}(\cdot,\cdot)$ be the pairing that we defined before using s (which is alg thanks to Lefschetz)

$$\tilde{Q}(\xi, \cdot) = \tilde{Q}([Z'], \cdot)$$ where Z' is an algebraic cycle on Y

(we are basically using again the perfect pairing on algebraic cycles)

This implies:

alg cycle on X

$$\tilde{Q}([z]-i_*[z'],[w]) = \tilde{Q}(i_*(\xi-[z']),[w]) =$$

$$= \tilde{Q}(\xi-[z'], i^*[w]) = 0$$

(projection formula $i_*(\alpha \smile i^*\beta) = i_*\alpha \smile \beta$)

because $\tilde{Q}(\xi,-) = \tilde{Q}([z'],-)$

$$\Downarrow$$

$$[z] \equiv i_*[z'] \Rightarrow [z] = i_*[z']$$

□.

hom vs num conj is implied by kf.

Lefschetz $\Rightarrow$ Künneth (recall δ_k is the id_{H^k} and 0_{H^j}, $j \neq k$)

$$\boxed{\text{Kleiman Formula: } \delta_k = \Lambda_{n-k} \circ \Big(1 - \sum_{j > 2n-k} \delta_j\Big) \circ L^{n-k} \circ \Big(1 - \sum_{i<k} \delta_i\Big)}$$

$\forall \alpha \in H^i$, $i<k$ is sent to 0

By Lefschetz, Λ_{n-k} is algebraic

We know that L^{n-k} is algebraic

$\Rightarrow$ by induction, δ_k is algebraic $\square$

Lemma

X sm proj verifies Lefschetz in deg k $\iff$ $\exists\, S$ sm proj of dim $l \geq k$ verifying Lefschetz in deg $\leq k-2$ and Z algebraic cycle on $S \times X$ such that

$$[Z]_* : H^{2l-k}(S) \longrightarrow H^k(X) \quad \text{surj}$$

Proof

is algebraic !

Consider: $[Z]_* \circ L^{\ell-k} \circ s \circ [Z]^* : H^{2n-k}(X) \to H^k(S) \to H^{2\ell-k}(S) \to H^k(X)$

If we show that this composition is an isomorphism, we're done. Actually, is enough to show injectivity

Suppose: $([Z]_* \circ L^{\ell-k} \circ s \circ [Z]^*)(\alpha) = 0 \,. \quad (\star)$

$$\Rightarrow \tilde{Q}([Z]^*\alpha, [Z]^*\beta) = 2^{k} \int_S L^{\ell-k} s [Z]^*\alpha \cup [Z]^*\beta$$

$\| \leftarrow$ proj formula

$0 \quad + \ \star$

for some $\beta \in H^{2n-k}(X)$

On the other hand, $\tilde{Q}$ is a non-degenerate pos. def pairing also when restricted $[Z]^* H^{2n-k}(X)$.

$\Rightarrow \quad [Z]^* \alpha = 0$

But $[Z]^* = {}^t[Z]_* \Rightarrow$ it's injective

$\Downarrow$

$\alpha = 0$

$\swarrow$

that composition is inj (the green one) $\square$

Proof of $V + K \Rightarrow L$

$Y_k :=$ intersection of $(n-k)$ hyperplanes in X

$H^k(Y_k) \to H^{2n-k}(X)$ surj by weak Lefschetz.

$H^k(Y_k) \otimes H^k(X) \to H^{2n-k}(X) \otimes H^k(X)$ surj.

$\Rightarrow \delta_{2n-k}$ (algebraic by K) is equal to $(i, id)_* [Z]$

w/ Z algebraic (here we use V)

$i_* \circ [Z]^* = id$ because $i_* \circ [Z]^* = [(i, id)_* Z]_* = \delta_{2n-k}$

on $H^{2n-k}(X)$

This implies that $[Z]^*$ injective

$\Rightarrow$ $[Z]_*$ is surjective

$\Downarrow$

we can apply the lemma that we proved before

$\hookrightarrow$ by induction, Lefschetz holds on X.

$\square$