Third lecture

(1) Decomposition of the Δ
(and how we can use it)

(2) Zero cycles

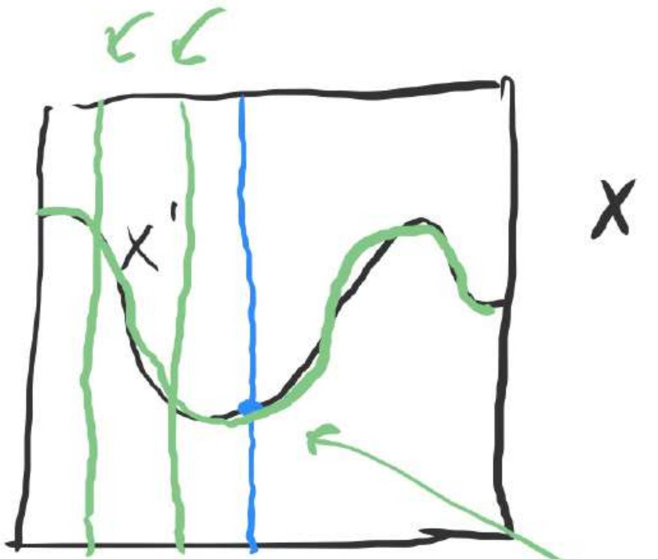
Decomposition of the Δ

Thm (Bloch-Srinivasa)

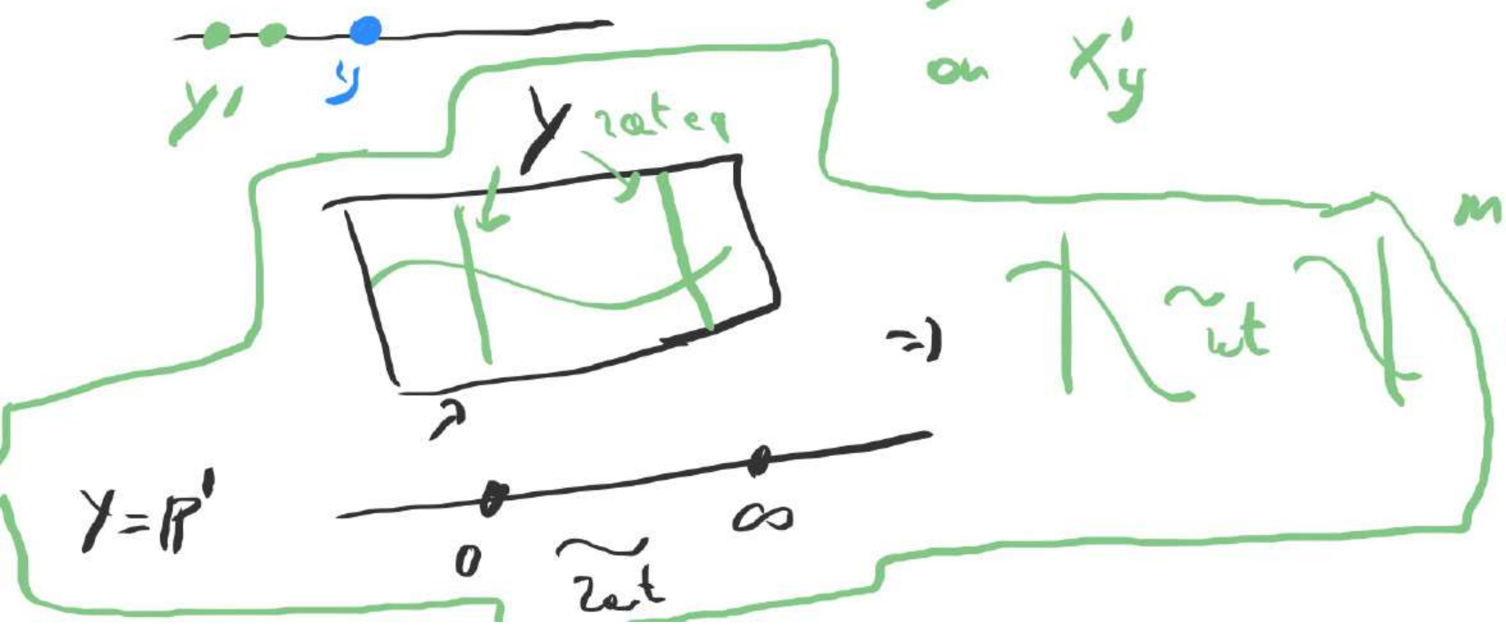
Let $f: X \to Y$ be a smooth morphism of smooth proj varieties. Suppose that $\exists X' \subsetneq X$ closed subvariety and a cycle Z such that $\forall y \in Y$ we have:

$$Z_y \in CH^*(X_y) \text{ is supported on } X'_y$$

Then $\exists Y' \subsetneq Y$ closed such that $mZ = Z_1 + Z_2$, $m \in \mathbb{Z}$, where Z_1 is supp. on X' and Z_2 is supp. on $f^{-1}(Y')$



Z_y is supported on X'_y



What can we say about Z?

⇒ Z will be of the form $Z_1 + Z_2$

Cor

Consider $X \times Y \xrightarrow{pr_2} Y$. Suppose $\exists X' \subsetneq X$ and cycle Z on $X \times Y$ st Z_y is supported on X' $\left([Z_y] = 0 \text{ in } CH^*(X \setminus X') \right)$

$\Rightarrow \exists Y' \subsetneq Y$ st $mZ = Z_1 + Z_2$ where

Z_1 is supported on $X' \times Y$
Z_2 is supported on $X \times Y'$

Rmk: in particular, we can consider $X \times X \xrightarrow{pr_1} X$ and $Z = \Delta_X$

Cor (dec of Δ)

Let X be sm proj var and suppose $\exists W \subsetneq X$ such that $CH_0(W) \longrightarrow CH_0(X)$ is surjective. Then

$$\boxed{m\Delta_X = Z_1 + Z_2}$$

where Z_1 supp. on $X \times W$

Z_2 supp on $T \times X$ for some $T \subsetneq X$

Sketch of proof

$\Delta_X|_{X_p} = [p] \in CH_0(X)$, hence $[p]$ is supported on W $\square$

$X_p \leftarrow p \in X$

Some applications

(1) Prop

X sm proj, $W \subsetneq X$ dim 3 st $CH_0(W) \twoheadrightarrow CH_0(X)$ surj

$\Rightarrow$ the Hodge conjecture is true for $H^4(X,\mathbb{Q})$.

Proof

We have to show: $\alpha \in H^{2,2}(X,\mathbb{Q}) \Rightarrow \alpha$ is algebraic

We have: $m\Delta_X = Z_1 + Z_2$. $\quad$ Z_1 cycle on $X \times W$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ Z_2 cycle on $T \times X$, $T \subsetneq X$

$$m\Delta_X = Z_1 + Z_2, \qquad Z_1 = (id, i)_* Z_1', \qquad Z_2 = (j, id)_* Z_2'$$

$i : W \hookrightarrow X$

$$m\Delta_X^*\alpha = \left[(id, i)_* Z_1'\right]^*\alpha + \left[(j, id)_* Z_2'\right]^*\alpha$$

$$\| \qquad\qquad // \qquad\qquad\qquad \|$$

$$m\alpha \qquad Z_1'^*(i^*\alpha) + j_*(Z_2'^*\alpha)$$

belongs to $H^{2,2}(W_1, \mathbb{Q})$ which is isomorphic to $L \cup H^{2,1}(W, \mathbb{Q})$ by Lefschetz theorem $\Rightarrow i^*\alpha = L \cup \beta$ w/ β alg. $\Rightarrow i^*\alpha$ alg.

belongs to $H^{1,1}(T, \mathbb{Q})$ $\Rightarrow Z_2'^*\alpha$ algebraic

$\Rightarrow$ α algebraic $\square$

Zero cycles

$Z_i(X) = \langle [V] \mid V \subset X \text{ of dim } i \rangle$

$$CH_i(X) = Z_i(X) / \sim_{rat}$$

rational equivalence relation

Here we focus on $CH_0(X)$ (0 cycles).

$$cl : CH_0(X)_{\mathbb{Q}} \longrightarrow H^{2n}(X, \mathbb{Q}) \cong \mathbb{Q}$$

$$\left[\sum p_i - \sum q_i\right] \longmapsto \deg\left[\sum p_i - \sum q_i\right]$$

degree 0 cycles

$p_i, q_i \in X$

Def: $CH_0(X)_{hom} = \ker(cl)$

We want to study $CH_0(X)_{hom}$

Eg let X be a curve, then $CH_0(X) = CH^1(X)$
$$\| \qquad\qquad\qquad\qquad \|$$
$$\Downarrow \qquad\qquad\qquad\qquad Pic(X)$$

$$CH_0(X)_{hom} \xrightarrow{\sim} Pic^0(X) = J(X)$$

jacobian of X

$$J(X) = \frac{H^{1,0}(X)^\vee}{H_1(X,\mathbb{Z})}$$

$CH_0(X)_{hom} \subset CH_0(X)$

The associated graded group

$H^2(X,\mathbb{Z}) \oplus J(X)$

Upshot: $CH_0(X)$ can be understood via Hodge theory

Q: does this generalize to higher dim. vars?
The next step is X surface.

$$\begin{array}{ccc} CH_0(X)_{hom} & \longrightarrow & H^0(X,\Omega^1)^\vee \big/ H_1(X,\mathbb{Z}) =: Alb(X) \\ \Psi & & \Psi \\ [p-q] & \longmapsto & \int_q^p \end{array}$$

$$Alb(X) = J^3(X)$$

3rd intermediate jacobian

$p, q \in X$

choose a path from q to p

$\omega \in H^0(X,\Omega^1)$ is sent to $\int_q^p \omega$

There are other characterizations of the property that alb_X is isomorphism

Prop TFAE:

(1) alb_X isomorphism

(2) $CH_0(X)$ is representable, i.e. $\exists m \geq 0$ such that

$$X^{(m)} \times X^{(m)} \longrightarrow CH_0(X)_{hom} \text{ is surjective}$$

$$\left(\sum p_i , \sum q_i\right) \longmapsto \left[\sum p_i - \sum q_i\right]$$

(3) $CH_0(X)$ finite dimensional

$CH_0(X)$ being finite dimensional means that:

$$X^{(m)} \times X^{(m)} \xrightarrow{\sigma_m} CH_0(X)_{hom} \Rightarrow$$ the general fiber is the union of alg subvars $\exists Z$ of maximal dim, say $r(m)$

Define $\dim(\mathrm{Im}\,\sigma_m) \overset{def}{=} 2m\cdot n - r(m)$

Then $CH_0(X)$ is infinite dimensional if

$$\lim_{m\to\infty} \dim(\mathrm{Im}\,\sigma_m) = +\infty$$

Thm (Mumford)

X surface

$$CH_0(X) \text{ representable} \Longrightarrow H^0(X, \Omega^2) = 0$$

$\Updownarrow$

alb_X isomorphism