Lecture 4 : zero cycles and Bloch conjecture

Recap

X sm proj var$/\mathbb{C}$, $CH_0(X)$ group of zero cycles

$\cup$

$CH_0(X)_{hom}$ group of zero cycles of degree 0

Ψ

$$\sum_{p_i \in X}[p_i] - \sum_{q_i \in X}[q_i]$$

abelian variety

$$CH_0(X)_{hom} \xrightarrow{alb_X} Alb_X = H^0(X,\Omega_X)^\vee / H_1(X,\mathbb{Z}) \leftarrow \gamma \mapsto \int_\gamma$$

$[p]-[q] \longmapsto {}_q\int^p$

Question: is $CH_0(X)_{hom} \longrightarrow Alb_X$ an isomorphism?

- if $\dim(X) = 1 \Rightarrow$ yes! $\left(CH_0(X)_{hom} = Pic^0_X = J_X \right)$
- if $\dim(X) = 2 \Rightarrow$?

(Why do we care? If alb_X is an iso, then "Hodge theory controls zero cycles")

$CH_0(X)_{hom} \subseteq CH_0(X)$ st

$CH_0(X) / CH_0(X)_{hom} = H^{top}(X, \mathbb{Z})$

$CH_0(X)_{hom} \xrightarrow{\cong} Alb_X = H^0(X, \Omega_X)^\vee / \Lambda$

Thm (Rotmann) TFAE

(i) alb_X isomorphism

(ii) $CH_0(X)$ finite dimensional

(iii) $CH_0(X)$ representable

(iv) $(\dim X = 2)$ $\exists C \subseteq X$ st $CH_0(C) \twoheadrightarrow CH_0(X)$ surj

$$X^{(m)} \times X^{(m)} \xrightarrow{6_m} CH_0(X)_{hom}$$

$$\sum p_i, \sum q_i \mapsto \sum([p_i]-[q_i])$$

$$\lim_{m\to+\infty} \dim(\mathrm{Im}\, 6_m) < K$$

$\exists m >> 0$

$$X^{(m)} \longrightarrow CH_0(X)_{hom}$$

$$\sum p_i \longmapsto \sum [p_i] - m[p_0]$$

surjective

Something we will not prove. (ii) $\Longleftrightarrow$ (iii)

Proof of (ii) $\Rightarrow$ (iv)

can assume equality

By hypothesis, $\lim_{m\to\infty} \dim \operatorname{Im} \sigma_m \leq K$

Fact: $\sigma_m^{-1}(p) = \bigcup Z_i$, Z_i algebraic

- $\sigma_m : X^{(m)} \to CH_0(X)_h$, $\sum p_i \longmapsto \sum [p_i] - m[p_0]$
- dim fiber of $\sigma_m \overset{\text{def}}{=} \max\left\{ \dim Z_i \;\middle|\; Z_i \subset \sigma_m^{-1}(p) \text{ irreducible component} \right\}$

general

- $\dim \operatorname{Im}(\sigma_m) = mn -$ dim fiber

$\Rightarrow \exists Z \subset \sigma_m^{-1}(p)$ of dim $mn - K$

Claim: $\nexists W \subset X^{(i)}$ of $\dim < i$ such that $Z \subseteq X^{(m-i)} + W$

Lemma if the claim is true $\Rightarrow \exists Y_1 \subseteq X$ ample divisor such that

$$Y_1^{(n)} \cap Z \neq \emptyset$$

for every p, $Z \subseteq \sigma_m^{-1}(p)$ maximal.

Proof of the claim

Suppose $Z \subseteq X^{(m-i)} + W$ for some $W \leq X^{(i)}$ of $\dim < i$

Consider $Z' = \{(z,w) \mid z+w \in Z\} \subseteq X^{(m-i)} \times W$

Consider $Z' \longrightarrow Z$ surjective $\Rightarrow \dim(Z') \geq mn - K$

$z,w \longmapsto z+w$

$\Downarrow$

Consider

$$\begin{array}{ccc} Z' & \supseteq & Z'_w \subseteq X^{(m-i)} \\ \downarrow & & \downarrow \\ W & \ni & w \end{array}$$

$\Rightarrow \boxed{\dim(Z'_w) \geq mn - K - i + 1}$

On the other hand:

$$\sigma_m(z+w) = \sigma_{m-i}(z) + \sigma_i(w)$$

$\sigma_m : X^{(m)} \to CH_0(X)_h$

Rmk: $\sigma_m|_Z$ constant bcs Z is a fiber

$\Rightarrow$ if we fix $w_0 \in W$

$\Rightarrow$ σ_{m-i} must be constant on Z'_{w_0}

But $\dim(Z'_{w_0}) \leq$ dimension of general fiber of $\sigma_{m-i} : X^{(m-i)} \longrightarrow CH_0(X)_{hom}$

$$\Rightarrow \boxed{\dim(Z'_{w_0}) \leq (m-i)n - K}$$

$\leftarrow$ compare this w/ the other one you get absurd

This ends the proof of the claim

By the lemma $\exists\, Y_1 \subseteq X$ such that

$\uparrow$ very ample $\quad Y_1^{(m)} \cap Z \neq \emptyset$

$\Rightarrow CH_0(Y_1)_{hom} \twoheadrightarrow CH_0(X)_{hom}$

Y_1 satisfies the hypothesis of finite dim

$\Rightarrow$ we can reiterate the argument!

$\Rightarrow C = Y_1 \cap \dots \cap Y_m$ st $CH_0(C) \twoheadrightarrow CH_0(X)$

$\square$

Proof of iv ⇒ i

$$\begin{tikzcd} CH_0(C)_{hom} \arrow[r, two heads] \arrow[d, "\cong"'] & CH_0(X)_{hom} \arrow[d] \\ Alb_C = J_C \arrow[r] & Alb_X \end{tikzcd}$$

$\Rightarrow$ alb_X is injective (alb_X always surj)

by the universal property of alb_X

$\Rightarrow$ $J_C \longrightarrow CH_0(X)_{hom}$ this is surj group hom

<u>Fact</u>: (1) $\exists\, \Gamma \subseteq J_C \times X$

$J_C \to CH_0(X)_{hom}$.

$x \mapsto \Gamma_*([x]-[x_0])$

(2) We can assume $\ker(\Gamma_*)$ countable

Consider $R \subseteq X \times J_C$ def. as $R = \left\{ (x,a) \;\middle|\; \Gamma_*(a) = [x] - [x_0] \right\}$

<u>Rmk</u>: $R \longrightarrow X$ surj. (this follows from Γ_* being surj)

here is where we are using the hyp.

$\Rightarrow \exists R_0 \to X$ alg. variety such that $R_0 \to X$ surj and finite of deg r.

$$\begin{tikzcd}
 & & X \arrow[d] \arrow[dr, "\mathrm{alb}_{J_C}^{-1} \circ R_{0*}"] & \\
J_C \arrow[r, "\Gamma_*"] \arrow[rrr, bend right] & CH_0(X)_h \arrow[r, "\mathrm{alb}_X"] & \mathrm{Alb}_X \arrow[r] & J_C
\end{tikzcd}$$

by universal property

the green arrow is $r \cdot \mathrm{Id}$

$\Downarrow$

$\ker(\mathrm{alb}_X)$ is torsion

This implies that $CH_0(X)_{hom} \otimes \mathbb{Q} \xrightarrow{\cong} Alb_X \otimes \mathbb{Q}$

interpreted as group

Thm (R)

alb_X is an isomorphism on torsion elements

$\Rightarrow CH_0(X)_{hom} \xrightarrow{\sim} Alb_X$

$\square$

$CH_0(X)_{hom} \cong Alb_X \Rightarrow CH_0(X)$ representable

$$\begin{tikzcd} X^{(m)} \arrow[d] \arrow[dr] & \\ CH_0(X)_{hom} \arrow[r, "\sim"] & Alb_X \arrow[l, bend left] \end{tikzcd}$$

we've seen last time that this is surj. for $m \gg 0$

$\square$

Thm (Mumford)

Let X be a surface and suppose that $CH_0(X)$ is representable/fd $\Rightarrow H^0(X, K_X) = 0$

$K_X = \Omega_X^2$

we know that this condition on zero cycles $\Leftrightarrow$ alb_X isom.

Proof

$\dim(W) < k \qquad \dim(X) = n > k$

if $\exists W \subsetneq X$ such that $CH_0(W) \to CH_0(X)$ surjective

$\Rightarrow \quad m\Delta_X = Z_1 + Z_2$

diagonal in $X \times X$

$= (id, i)_* Z_1'$

$Z_1' \subseteq X \times W$

$= (j, id)_* Z_2'$

$Z_2' \subseteq T \times X$

$i : W \hookrightarrow X$

$j : T \hookrightarrow X$

Pick $\alpha \in H^0(X, \Omega^k_X) = H^{k,0}(X) \subseteq H^k(X, \mathbb{C})$

$m\alpha = m\Delta_X^* \alpha = Z_1^* \alpha + Z_2^* \alpha = \cancel{Z_1^*(i^*\alpha)} + j_* Z_2^* \alpha$

$i^*\alpha = 0$ because $H^0(W, \Omega^k) = 0$

j_* as morphism of HS has at least deg $(1,1)$

$\Leftarrow$ $j_*(\cdot)$ is $(k,0)$ must be 0

This shows that

If $\exists W \subseteq X$ of dim $< k \Rightarrow H^0(X, \Omega^i_X) = 0 \quad i > k$

In particular if $\exists C \subseteq X$ surface st $CH_0(C) \twoheadrightarrow CH_0(X)$
curve

then $H^0(X, \Omega^2_X) = 0$

But we've proved that $CH_0(C) \twoheadrightarrow CH_0(X)$

is equivalent to $CH_0(X)$ being rep / fd.

$\square$