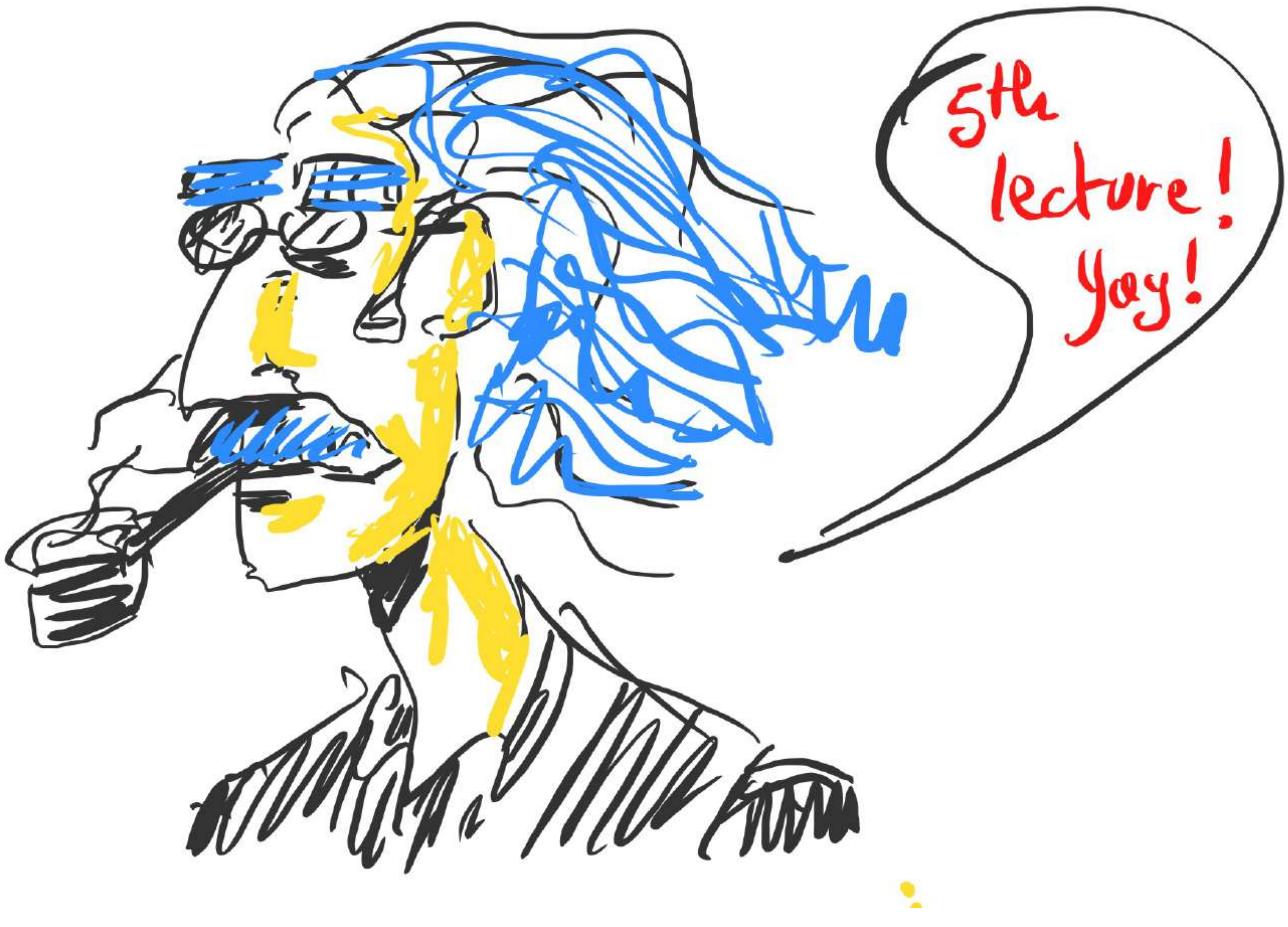
5th lecture! Yay!

Plan

Main goal: proof of Bloch conj for surfaces
NOT of general type

(1) Recap on classification of surfaces

(2) The proof

Recap

$\boxed{\text{Bloch conj}}$: X sm proj surface w/ $h^0(\Omega^2) =: p_g = 0$ geometric genus

$\Longrightarrow$ $CH_0(X)$ is finite dimensional representable

$alb_X : CH_0(X)_{hom} \longrightarrow Alb_X$ is an iso (injective)

$\exists C \subseteq X$ ample such that

$CH_0(C) \twoheadrightarrow CH_0(X)$

Recap on classification of surfaces / $\mathbb{C}$

Classification up to <u>birational equivalences</u>

It's based on the Kodaira dimension.

canonical ring

<u>Lemma</u>

$$\bigoplus_{m \geq 0} H^0(X, K_X^{\otimes m})$$ is a birational invariant

(actually, $H^0(X, \Omega_X^p)$ is bir. inv)

canonical bundle

<u>Sketch of proof</u>

The proof is based on Hartogs theorem. $\square$

$$X \dashrightarrow \mathrm{Proj}\left(\bigoplus_{m\geq 0} H^0(X, K_X^{\otimes m})\right) = X_{can}$$

Def

$$\mathrm{kod}(X) = \dim(X_{can}) \qquad (\dim(\emptyset) = -\infty)$$

There is another definition:

$\mathrm{kod}(X)$ satisfies $\displaystyle\lim_{m\to+\infty} \frac{h^0(X, K_X^{\otimes m})}{m^{\mathrm{kod}(X)}} < +\infty$

In particular $\mathrm{kod}(X) \in \{-\infty, 0, 1, \dots, \dim(X)\}$

For X surface, $\mathrm{kod}(X) \in \{-\infty, 0, 1, 2\}$

Rmk

$H^0(X, \Omega^2_X) = H^0(X, K_X)$ is a birational invariant,

but also $CH_0(X)$ is a birational invariant!

(**Sketch of proof**:

$$\begin{tikzcd} & X_1 \arrow[dl] \arrow[r] & X_2 & X_3 \arrow[l] \arrow[r] & \cdots \\ X \arrow[rrrr, dashed, leftrightarrow] &&&& Y \end{tikzcd}$$

$\leftarrow$ all of these are blow-ups and blow-downs

$\Downarrow$

it's enough to prove that $CH_0(X)$ is not affected by blow-ups and blow-downs $\square$)

$\Downarrow$

it's enough to verify the Bloch conj up to bir. eq

$\operatorname{kod}(X) = -\infty$

Thm (Beauville)

If X has K-dim $= -\infty \Rightarrow X \leftarrow \cdots \rightarrow X'$ birationally ruled

$$\begin{array}{ccc} X' & \supseteq & \mathbb{P}^1 \\ \downarrow & & \downarrow \\ C & \ni & p \end{array}$$

$\mathbb{P}^1$ cover the whole X'

Let us compute $H^0(X', K_{X'})$

Adjunction formula:

$N_{\mathbb{P}^1} = \mathcal{O}$

$$K_{X'}^{\otimes m}\big|_{\mathbb{P}^1} = K_{\mathbb{P}^1}^{\otimes m} + \det(N_{\mathbb{P}^1})^{\otimes m} = \mathcal{O}_{\mathbb{P}^1}(-2)^{\otimes m} \Rightarrow H^0(X', K_{X'}^{\otimes m}) = 0$$

We expect by Bloch conjecture that $CH_0(X')$ is finite dim'

Kod(X) = −∞

Thm (Beauville)

If X has K-dim $= -\infty \Rightarrow X \leftarrow \dots \rightarrow X'$ birationally ruled

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We expect by Bloch conjecture that $CH_0(X')$ is finite dim

Proof of the Bloch Conj for $\mathrm{Kod}(X) = -\infty$

$x \in X$ birationally ruled $\Rightarrow$
$$\begin{array}{ccccc} x \in & \mathbb{P}^1 & \subset & X \\ & \downarrow & & \downarrow \\ & p & \in & C \end{array}$$

Consider $D \subseteq X$

very ample divisor $\Rightarrow$ $D \cap \mathbb{P}^1 \supseteq \{y\cdot\}$, moreover

$x \sim_{rat} y$ because we are on $\mathbb{P}^1$

$\Downarrow$

$$CH_0(D) \twoheadrightarrow CH_0(X)$$

and we've seen that this is equivalent to finite dimensionality of $CH_0(X)$. $\square$

irregularity.

Lemma

If X sm proj surf has $p_g = 0 \Rightarrow q(X) = h^1(X, \mathcal{O}_X) \leq 1$

Proof

Claim: $c_1(K_X)^2 = 0$

Given the claim, we can apply Noether's formula:

$$1 - q + \cancel{p_g} = \chi(X) = \frac{\cancel{c_1(K_X)^2} + c_2(T_X)}{12} = \frac{1}{12}\left(\chi_{top}(X)\right)$$

$$= \frac{1}{12}\left(\sum (-1)^i b_i\right) = \frac{1}{12}\left(1 - 2q + h^{1,1} - 2q + 1\right)$$

we have $b_i = \sum h^{p,q}$ for $p+q=i$

$$1 - q = \frac{1}{12}\left(2 - 4q + h^{1,1}\right)$$

$$0 = -12 + 12q + 2 - 4q + h^{1,1} = -10 + 8q + h^{1,1}$$

$$q = \frac{1}{8}\left(10 - h^{1,1}\right) < 2 \quad \square$$

Theorem

$\mathrm{kod}(X)=0 \Rightarrow$

(i) X ~~abelian surface~~ ← we don't care ($p_g \neq 0$, $q=2$)

(ii) X ~~K3 surface~~ ← $p_g \neq 0$

(iii) X Enriques (quotient of a K3 by an involution which has no fixed pts)

we should take care of only these 2 cases

(iv) $X = E \times F/G$ where:

E elliptic curve, F/G rational curve

$G \subseteq \mathrm{Aut}(E) \times \mathrm{Aut}(F)$ and acts on E by translations

Proof of the Bloch conj

Suppose $g(X)=1 \Rightarrow$ (a) $\mathrm{Kod}(X)=0$, $X = E\times F/G$

(b1) $\mathrm{Kod}(X)=1$, $X \to \mathrm{Alb}_X = E$ isotrivial w/ fibers of $g>1$

(b2) $\mathrm{Kod}(X)=1$, $X \to E$ elliptic fibration

Σ

$J \to E$

Claim all these cases can be reduced to case (a).

How to reduce (b1) to (a)?

$$\begin{array}{ccc} \tilde{E}\times C & \to & X \\ \downarrow & & \downarrow \\ \tilde{E} & \xrightarrow[\text{cover}]{\text{étale}} & E \end{array} \Rightarrow X = \tilde{E}\times C/G$$

by isotriviality of the fibration

It is true that $C/G \cong \mathbb{P}^1$? It's enough to show that:

$$\underset{\overset{\|}{0}}{H^0(C/G, \Omega_{C/G})} = H^0(C, \Omega_C)^G$$

Observe that G acts on $\tilde{E}$ by translations, hence

$$H^0(\tilde{E}, \Omega_{\tilde{E}})^G = H^0(\tilde{E}, \Omega_{\tilde{E}})$$

But: $\quad 0 = H^0(X, K_X) = H^0(\tilde{E} \times C, \Omega_{\tilde{E}} \otimes \Omega_C)^G$

If $\quad H^0(C, \Omega_C)^G \neq 0 \Rightarrow H^0(\tilde{E} \times C, \Omega_{\tilde{E}} \otimes \Omega_C) \neq 0$

Hence $\quad H^0(C, \Omega_C)^G = 0 \Rightarrow C/G \cong \mathbb{P}^1$ $\Leftarrow$ this shows (b2) $\Rightarrow$ (a)

How can we reduce (b2) to (a)?

$X \downarrow E$ Suppose to have a section $E \xrightarrow{\sigma} X$

$\Rightarrow X \longrightarrow J$

$x \longmapsto \mathrm{alb}_{X_\eta}([x]-[\sigma])$

If you don't have a section, consider $\exists C \subseteq X$ such that

$$\begin{tikzcd} X_C \arrow[r] \arrow[d] & X \arrow[d] \\ C \arrow[r] \arrow[u, bend left, "\exists\sigma"] & E \end{tikzcd}$$

<u>Ex</u>: show that if you can prove Bloch conjecture for J_C then it's true for X_C which implies BC for X.

We're in the setup (a) : $X = E \times F/G$ w/ $G \subseteq Aut(E) \times Aut(F)$ finite

G acts via traslations on E, $F/G \cong \mathbb{P}^1$

Claim 1 $CH_0(F)^G_{\mathbb{Q}} \cong \mathbb{Q} \cdot [s_0]$

Proof $CH_0(F)^G_{\mathbb{Q}} \cong CH_0(F/G)_{\mathbb{Q}} \cong CH_0(\mathbb{P}^1)_{\mathbb{Q}}$

Claim 2 $CH_0(E)^G_{\mathbb{Q}} \cong CH_0(E)_{\mathbb{Q}}$

Proof : $G \overset{\text{triv}}{\curvearrowright} CH_0(E)/CH_0(E)_{hom}$ because the latter is $\cong \mathbb{Z}$

$G \overset{\text{triv}}{\curvearrowright} CH_0(E)_{hom}$ because $G \overset{\text{triv}}{\curvearrowright} H^0(E, \Omega_E)$

$\Downarrow$

$G \overset{\text{triv}}{\curvearrowright} Alb_E \cong CH_0(E)_{hom}$

$\Rightarrow$ let $z \in CH_0(E)$, $g^*z = z + z_g \Rightarrow g^*g^*z = z + 2z_g$

$\uparrow$ hom. trivial

$\exists n$ st $(g^*)^n z = z + n z_g \Rightarrow n z_g = 0$

$(g^*)^n z = z = id^* z$

or, in other terms, the action of G on $CH_0(E)_{\mathbb{Q}}$ is trivial $\square$

Consider $(e, c) \in E \times F$

$$\sum_{g \in G} (g_1^* e, g_2^* c) = \sum_{g \in G} (e, g_2^* c) = \left(e, \sum_{g \in G} g_2^* c\right)$$

$g_1^* = id$ up to torsion

this is invariant w/ G-action but $F/G = \mathbb{P}^1$ and we know $CH_0(\mathbb{P}^1)_{\mathbb{Q}} = \mathbb{Q}$

$$= (e, n c_0)$$

some fixed $c_0 \in \mathbb{P}^1$

In other terms, we have shown that this is surjective

$$CH_0(E\times\{c_0\})_{\mathbb{Q}} = CH_0(E\times\{c_0\})^G_{\mathbb{Q}} \twoheadrightarrow CH_0(E\times F)^G_{\mathbb{Q}} = CH_0(X)_{\mathbb{Q}}$$

This implies that

$$CH_0(X)_{hom}\otimes\mathbb{Q} \xrightarrow{\sim} \mathrm{Alb}_X\otimes\mathbb{Q}$$

By Roitman's thm, we know that $CH_0(X)^{tors}_{hom} \xrightarrow{\sim} \mathrm{Alb}_X^{tors}$

$\Rightarrow CH_0(X)_h \xrightarrow{\sim} \mathrm{Alb}_X \Rightarrow CH_0(X)$ finite dim./rep.

$$q(X) = 0 \Rightarrow \begin{array}{c} X \\ \downarrow \\ \mathbb{P}^1 \end{array} \text{ w/ elliptic fibres} \xRightarrow{\text{blow-up}} \begin{array}{c} \tilde{X} \\ \downarrow \\ \mathbb{P}^1 \end{array} \text{ w/ smooth elliptic fibres}$$

$$\xRightarrow{\text{jacobian fibration}} \begin{array}{c} J \\ \downarrow \\ \mathbb{P}^1 \end{array}$$

we've reduced ourselves to study J

consider again

$$\tilde{X} \longrightarrow J$$
$$x \longmapsto \mathrm{alb}_{x_p}([x] - [G(p)])$$

$$\Rightarrow CH_0(J)_{\mathbb{Q}} \cong CH_0(\tilde{X})_{\mathbb{Q}}$$
$$\Downarrow$$
$$CH_0(J) \cong CH_0(\tilde{X})$$

($q=0 \Rightarrow$ Alb is trivial $\Rightarrow$ no torsion pts for CH_0)

Prop

$q(X), P_g(X) = 0$
$\Rightarrow P_g(J) = 0$
$q(J) = 0$
$P_2(J) = 0$