
6th lecture!

Topic for the next 2 lectures: integral Hodge conj and rationality problems

Plan

(1) The integral Hodge conjecture is false (Kollár's example)

(2) Still, there is some hope

(3) Actually, in some cases there is really no hope that IHC could be true

(4) In some other cases, there is some hope

Kollár's example

Integral Hodge conjecture is false
(∃ Hodge class α in $H^{p,p}(X)$ st α is not algebraic)

$Y \subseteq \mathbb{P}^4$ hypersurface in $\mathbb{P}^4$ of degree s.

Let $p \geq 9$ be an integer coprime with 6.

Consider $\mathbb{P}^4 \longrightarrow \mathbb{P}^4$ defined via:

for $\phi_0 \dots \phi_4$ general this is well def

$$\phi_0, \dots, \phi_4 \in \Gamma(\mathcal{O}(p)) \rightsquigarrow [x_0 : \dots : x_4] \longmapsto [\phi_0(\underline{x}) : \dots : \phi_4(\underline{x})]$$

Claim For general $\varphi_0, \dots, \varphi_4$ we have:

- $f: \mathbb{P}^4 \to \mathbb{P}^4$ has degree p^4 see exercises
- $f_Y : Y \longrightarrow f(Y) =: X_0$ is generically 1:1 not so easy to prove. start from exercise 1.

  It's not 1:1 on $\underset{\text{Codim } 1}{S \subset Y}$, and $f_S : S \to f(S)$ is generically 2:1. Moreover, f_S it's not 2:1 on $\underset{\text{Codim } 1}{D \subseteq S}$ and $f_D : D \to f(D)$ is 3:1.

Claim: $\deg(X_0) = p^3 s$

$$[X_0] = f_*[Y] = f_*(s[H]) = p^3 s\,[H]$$

$$\Uparrow$$

Proof: $f_*[\mathbb{P}^4] = p^4[\mathbb{P}^4]$;

$$f_* p[H] = f_* f^*[H] = [H] \cdot f_*[\mathbb{P}^4] = p^4[H] \Rightarrow f_*[H] = p^3[H]$$

Rmk: $f_*[H]^i = p^{4-i}[H]^i$

Claim

$\mathbb{P}H^0(\mathcal{O}(p^3s)) = \mathbb{P}^N \ni X$ is very general if belongs to $\mathbb{P}^N \setminus$ countable union of closed subvar.

Let X be a very general hypersurface in $\mathbb{P}^4$ of degree p^3s and let $C \subseteq X$ be a curve. Then

$$(X, C) \underset{def}{\sim} (X_0, C_0)$$

what do you mean?

$\exists U$ and $(\mathcal{X}, \mathcal{C})$

$$\begin{tikzcd}(\mathcal{X},\mathcal{C}) \arrow[d] \\ U\end{tikzcd}$$

and $u_0, u_1 \in U$

the fiber of $(\mathcal{X}, \mathcal{C})$ over u_0 is (X_0, C_0)

the fiber $/ u_1$ is (X, C)

Sketch of proof

$\mathrm{Hilb}_{\mathcal{X}/\mathbb{P}^N}$ is the Hilbert scheme of $(D \underset{curve}{\subseteq} Z \underset{p^3s}{\subseteq} \mathbb{P}^4)$ -

We would like to connect $(C \subseteq X \subseteq \mathbb{P}^4) \rightsquigarrow (C_0 \subseteq X_0 \subseteq \mathbb{P}^4)$

Problem: the geometry of $\mathrm{Hilb}_{\mathcal{X}/\mathbb{P}^N}$ can be very complicated, but $\exists\, \mathrm{Hilb}_{\mathcal{X}/\mathbb{P}^N} \longrightarrow \mathbb{P}^N$ given by $(D \subseteq Z \subseteq \mathbb{P}^4) \mapsto (Z \subseteq \mathbb{P}^4)$

$$\mathrm{Hilb}^{\mathrm{curve}}_{\mathcal{X}/\mathbb{P}^4} = \bigcup \mathrm{Hilb}^{\mathrm{curve},\nu}_{\mathcal{X}/\mathbb{P}^4}$$ ← Hilbert polynomial of the curve

irreducible!

Consider $\mathrm{Hilb}^{\mathrm{curve},\nu}_{\mathcal{X}/\mathbb{P}^4} \xrightarrow{\rho^\nu} \mathbb{P}^N \Rightarrow \mathbb{P}^N \setminus \bigcup \rho^\nu(\mathrm{Hilb}^{\mathrm{curve},\nu}_{\mathcal{X}/\mathbb{P}^N})$

ρ^ν is not dominant

this is a union over a countable set

by construction these are closed sets.

Consider $X \in \mathbb{P}^N \setminus \bigcup \cdots\cdots$

(hence X very general)

by construction (X, C) and (X_0, C_0) belong to an open subset in the same irreducible component of $\mathrm{Hilb}^{\mathrm{curve}}_{\mathcal{X}/\mathbb{P}^4} \Rightarrow$ can deform $(X, C) \rightsquigarrow (X_0, C_0)$

Upshot: $X \subset \mathbb{P}^4$ of degree $p^3 5$, $C \subseteq X$

$$(X, C) \underset{\text{def}}{\rightsquigarrow} (X_0, C_0) \longleftarrow$$ i'm saying $\exists C_0 \subseteq X_0$ st $(X, C) \rightsquigarrow (X_0, C_0)$

Important remark: $\deg(C_0) = \deg(C)$

What can we say about $\deg(C_0)$?

Rember: $f_Y : Y \to X_0$ gen $1:1$, $2:1$ on S, $3:1$ D

$\deg(C_0) = ?$ $\quad [C_0] \in \{ f_*[C_0'], 2 f_*[S_0'], 3 f_*[\overset{\uparrow}{D}] \}$

$$\Rightarrow \quad \deg(6[C_0]) = \deg(f_*\gamma) = \int_{X_0} f_*\gamma \cdot [H] = \int_Y \gamma \cdot f^*[H] \underset{y}{=} \int \gamma \cdot p[H]$$

Rmks

(1) Key point: every $C \subseteq X$ has degree p^t.

$$\Downarrow$$

α is not algebraic

(α would like to be $[\ell]$, ℓ line)

A natural question: what if X is rationally connected? Is the IHC true?

↑ every two points are connected by a line

Answer: no.

Proof: Consider $X \subset \mathbb{P}^{n+1}$ as the ones considered before (very general, of degree p^3)

$$X \subset \mathbb{P}^{n+1} \subset \mathbb{P}^{n+\ell} \Rightarrow \widetilde{\mathbb{P}}^{n+\ell} := Bl_X \mathbb{P}^{n+\ell}$$ is rationally connected.

In general, for $\tilde{Y} = Bl_Z Y$ we have:

$$H^k(\tilde{Y}) = \frac{H^k(Y) \oplus H^{k-2}(E)}{H^{k-2c}(Z)}$$

E is the exceptional divisor, in particular

$$\begin{tikzcd} E = \mathbb{P}(N_{Z,Y}) \arrow[d] \\ Z \end{tikzcd}$$

$c = \text{codim}(Z, Y)$

$$\begin{tikzcd} H^{k-4}(E) \; \xi^{-1}_{\cup} g^*\alpha \arrow[r, mapsto] & f^* i_*\alpha \in H^k(\tilde{Y}) \\ H^{k-2c}(Z) \ni \alpha \arrow[u, mapsto] \arrow[r, mapsto] & i_*\alpha \in H^k(Y) \arrow[u, mapsto] \end{tikzcd}$$

$$H^{2i}(\tilde{\mathbb{P}}^{n+l}) = H^{2i}(\mathbb{P}^{n+l}) \oplus \left(\bigoplus_{j=0}^{l-2} H^{2i-2j-2}(X) \cdot t^j \right)$$

$(i_{E*}(t^{i-n} \smile \alpha))$ is not algebraic.
$\in H^{2n-2}(X)$

is enough
$t^{i-n} \cup \alpha$ is not alg
$\Updownarrow$
α being not alg.

always true
for projective bdl

We have constructe a rationally connected variety Y such that $\exists \beta \in H^{2i}(Y)$ which is Hodge (α is (i,i)-form) but it's not algebraic.

If $\mathbb{Q}HC$ is true the $\mathcal{Z}^4$ and $\mathcal{Z}^{2n-2}$ are always torsion

(2) The previous construction works $i \neq 2, n-1$.

Actually:

$$\mathcal{Z}^4(X) \underset{def}{:=} Hdg^4(X) / H^4(X,\mathbb{Z})_{alg}$$

$$\mathcal{Z}^{2n-2}(X) \underset{def}{:=} Hdg^{2n-2}(X) / H^{2n-2}(X,\mathbb{Z})_{alg}$$

(IHC in deg $4, 2n-2$) $\iff \mathcal{Z}^4(X) = 0$ a $\mathcal{Z}^{2n-2}(X) = 0$

$$H^4(Bl_Z X) = H^4(X) \oplus H^2(E)$$

↖ by Lefschutz theorem

$$Hdg^2(E) = H^2(E)_{alg}$$

$$\cup$$

$$\underbrace{Hdg^4(Bl_Z X)/_{H^4(Bl_Z X)_{alg}}}_{Z^4(Bl_Z X)} = \underbrace{Hdg^4(X)/_{H^4(X)_{alg}}}_{Z^4(X)} \oplus \underbrace{Hdg^2(E)/_{H^2(E)_{alg}}}_{= 0 \text{ by Lefsclatz}}$$

$\square$

Upshot 3: $Z^4(X), Z^{2n-4}(X)$ birational invariants.

Natural Question: $\exists X$ not conn st $Z^4(X), Z^{2n-4}(X) \neq 0$?