
7th (last!) lecture

Recap

Integral Hodge conjecture

$\mathrm{Hdg}^{2k}(X,\mathbb{Q})$ rational Hodge classes

$H^{k,k}(X,\mathbb{C}) \cap H^{2k}(X,\mathbb{Q})$

Remark: torsion classes in $H^{2k}(X,\mathbb{Z})$ are by definition Hodge classes

Consider: $H^{2k}(X,\mathbb{Z}) \xrightarrow{f} H^{2k}(X,\mathbb{Q})$

Define: $\mathrm{Hdg}^{2k}(X,\mathbb{Z}) := f^{-1}(\mathrm{Hdg}^{2k}(X,\mathbb{Q}))$

IHC: $\mathrm{Hdg}^{2k}(X,\mathbb{Z}) / H^{2k}(X,\mathbb{Z})_{alg} =: Z^{2k}(X) = 0$

Q: $Z^4(X) = 0$ for rationally connected vars?

is a birational invariant!

What we will see in this lecture

Main goal: to show that $\exists X$ sm proj var unirational such that $Z^4(X) \neq 0$.

How: (1) Relate $Z^4(X)$ w/ another birational invariant

(2) Construct X w/ $H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}) \neq 0$

Unramified cohomology

let X_{cl} be X w/ euclidean topology
let X_{Zar} be X w/ Zariski topology. then

$$f : X_{cl} \longrightarrow X_{Zar}$$

is continous.

Let A be an abelian group, regarded as const. sheaf

$$\mathcal{H}^i(A) := R^i f_* A$$

Def

$$H^k_{nr}(X,A) := H^0(X_{Zar}, \mathcal{H}^k(A))$$

There is a "concrete" way to compute this cohomology

let $D \subseteq X$ be a closed subvariety and define

$$H^i(\mathbb{C}(D), A) := \lim_{\substack{U \subseteq D \\ \text{open}}} H^i(U, A)$$

$$U \overset{j}{\hookrightarrow} D$$

$$H^i(D) \xrightarrow{j^*} H^i(U)$$

$D \supseteq U_1 \supseteq U_2 \supseteq U_3$ (open, open)

$$H^i(D,A) \to H^i(U_1,A) \to H^i(U_2,A) \to \cdots$$

$$0 \to \mathcal{H}^i(A) \to H^i(\mathbb{C}(X), A) \xrightarrow{\text{res}} \bigoplus_{\substack{D \subseteq X \\ \text{codim } 1 \\ \text{subvar}}} H^{i-1}(\mathbb{C}(D), A)$$

$$\to \bigoplus_{\substack{D \subseteq X \\ \text{codim } 2 \\ \text{subvar}}} H^{i-2}(\mathbb{C}(D), A) \to \cdots$$

Thm (Bloch-Ogus)

The sequence above is a resolution for the sheaf $\mathcal{H}^i(A)$

$$\Rightarrow H^i_{nr}(X, A) = \ker\left(H^i(\mathbb{C}(X), A) \xrightarrow{\text{res}} \bigoplus_{D \in X^{(1)}} H^{i-1}(\mathbb{C}(D), A) \right)$$

Corollary

$H^i_{nr}(X, A)$ is a birational invariant

if $\mathbb{Q}$ HC is true
then $\mathrm{Tors}(Z^4(X))$
$\parallel$
$Z^4(X)$

Thm (CT - Voisin)

X sm proj var $\Rightarrow 0 \to H^3_{nr}(X,\mathbb{Z}) \otimes \mathbb{Q}/\mathbb{Z} \to H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}) \longrightarrow \mathrm{Tors}\left(Z^4(X)\right) \to 0$

$\mathbb{Q}$: Why we care?

Suppose X unirational $\Rightarrow CH_0(X) = \mathbb{Z}$ supported on one point.

This implies: (1) $\mathrm{Tors}(Z^4(X)) = Z^4(X)$

this follows from $\mathbb{Q}$HC in deg 4 for X uni

(2) $H^3_{nr}(X, \mathbb{Z}) = 0$

Proof

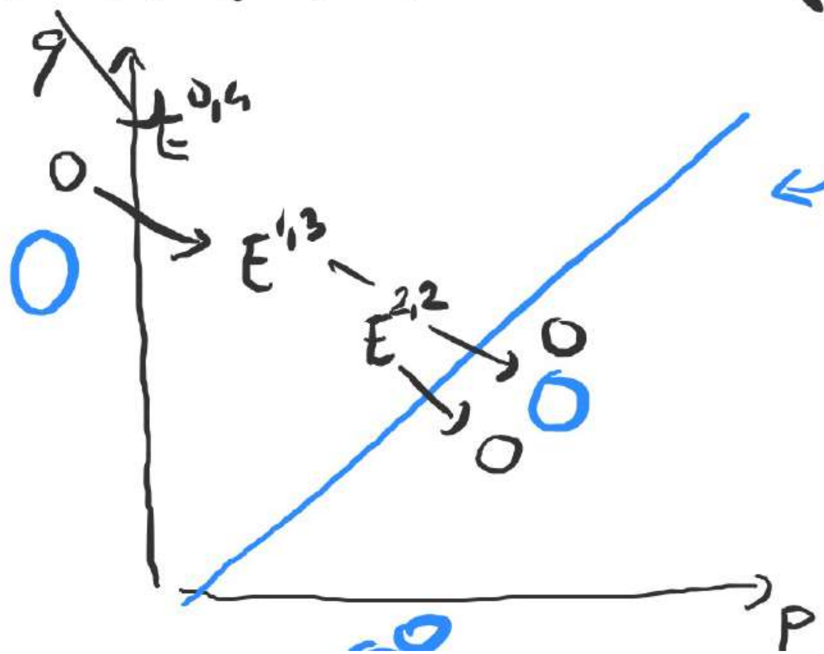
algebraic classes

Main thing: relate $Z^q(X)$ w/ H^*_{nr}

quite mysterious groups

Key: Leray spectral seq
- Bloch
  - Ogus

$$H^p(X, \mathcal{H}^q(\mathbb{Z})) = E_2^{p,q} \Rightarrow H^{p+q}(X, \mathbb{Z})$$



Thm
$H^p(X, \mathcal{H}^q(A)) = 0$ for $p > q$

$\Rightarrow$

$$\boxed{\begin{aligned} E_2^{2,2} &\twoheadrightarrow E_\infty^{2,2} \\ E_2^{1,3} &\cong E_\infty^{1,3} \\ E_2^{0,4} &\subset E_\infty^{0,4} \end{aligned}}$$

$$E_2^{2,2} = H^2(X, \mathcal{H}^2(\mathbb{Z})) \twoheadrightarrow E_\infty^{2,2} \subset H^4(X,\mathbb{Z})$$

$$\cong \; CH^2(X)/_{alg}$$

follows from Gersten resolution

lower piece of the filtration

this map is exactly cycle class map

$$\Rightarrow E_\infty^{2,2} = H^4(X,\mathbb{Z})_{alg}$$

We know from sp. seq that

$$\cong E_\infty^{4,3} = H^?(X, \mathcal{H}^3(\mathbb{Z}))$$

$$\mathcal{Z}^4(X) = H^4(X,\mathbb{Z})/E_\infty^{2,2} \supset N^1 \frac{H^4(X,\mathbb{Z})}{E_\infty^{2,2}} \Rightarrow \boxed{\begin{matrix} \mathrm{Tors}(\mathcal{Z}^4(X)) \\ \cong \\ \mathrm{Tors}(H^?(X, \mathcal{H}^3(\mathbb{Z}))) \end{matrix}}$$

the quot. is $E_\infty^{0,4} \in H^0(X, \mathcal{H}^4(\mathbb{Z}))$ which is torsion free!

By def of $\mathcal{H}^i(-)$, the s.e.q. $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ of const. sheaves on X_d induces:

$$\to \mathcal{H}^{i-1}(\mathbb{Q}/\mathbb{Z}) \to \mathcal{H}^i(\mathbb{Z}) \to \mathcal{H}^i(\mathbb{Q}) \to \mathcal{H}^i(\mathbb{Q}/\mathbb{Z}) \to \mathcal{H}^{i+1}(\mathbb{Z}) \to$$

Thm (B-O)

$\mathcal{H}^i(\mathbb{Z})$ is torsion free. In particular the following is exact

$$0 \to \mathcal{H}^i(\mathbb{Z}) \to \mathcal{H}^i(\mathbb{Q}) \to \mathcal{H}^i(\mathbb{Q}/\mathbb{Z}) \to 0$$

We can use it to compute $\mathrm{Tors}(H^1(X, \mathcal{H}^3(\mathbb{Z})))$!

$$0 \to H^0(X, \mathcal{H}^3(\mathbb{Z})) \to H^0(X, \mathcal{H}^3(\mathbb{Q})) \to H^0(X, \mathcal{H}^3(\mathbb{Q}/\mathbb{Z})) \to H^1(X, \mathcal{H}^3(\mathbb{Z}))$$

$$\mathrm{Tors}(H^1(X, \mathcal{H}^3(\mathbb{Z}))) = H^0(X, \mathcal{H}^3(\mathbb{Q}/\mathbb{Z})) \Big/ \left(H^0(X, \mathcal{H}^3(\mathbb{Z})) \otimes \mathbb{Q}/\mathbb{Z} \right) \qquad \square$$

Upshot

X unirational $\Rightarrow$ $Z^4(X) \cong H^3_{nr}(X, \mathbb{Q}/\mathbb{Z})$

CT-Ojanguren sixfold

X proj flat
$\downarrow$ the generic
B fiber is smooth quadric

Idea: use the theory of quadratic forms
$\Rightarrow$ construct interesting quadric bundles

1st ingredient: Pfister forms

$q, q' : V \to k$ quadratic forms $\Rightarrow$ $q \otimes q : V \otimes V \longrightarrow k$ quadratic form on $V \otimes V$

$\langle 1, -a \rangle = \begin{pmatrix} 1 & \\ & -a \end{pmatrix} = x^2 - aY^2$ (in general $\langle a, b, c, \ldots \rangle = \begin{pmatrix} a & & 0 \\ & b & \\ & & c \\ 0 & & & \ddots \end{pmatrix}$)

$\langle\langle a_1, \ldots, a_n \rangle\rangle = \langle 1, -a_1 \rangle \otimes \langle 1, -a_2 \rangle \otimes \cdots \otimes \langle 1, -a_n \rangle$

This quadratic form is called a Pfister forms

$a_1, a_2, a_3 \in k(x, y, z)$

Eg

$\langle\langle a_1, a_2, a_3 \rangle\rangle = \ldots = \langle 1, -a_1, -a_2, a_1 a_2, a_3, \ldots \rangle$

We can consider the subform $\langle 1, -a_1, -a_2, a_1 a_2, a_3 \rangle$

$$X = \{ X_0^2 + a_1 X_1^2 + a_2 X_2^2 + a_1 a_2 X_3^2 + a_3 X_4^2 = 0 \} \subseteq \mathbb{P}^3 \times \mathbb{P}^4$$

$\pi \downarrow$

$\mathbb{P}^3$ quadric bundle!

generic point of $\mathbb{P}^3$ is $\mathrm{Spec}(k(x,y,z))$

$X_{\mathbb{P}^3_{gen}} \subseteq \mathbb{P}^4_{k(x,y,z)}$

Thm (Arason)

$$\mathrm{Ker}\left(\pi^*: H^3_{et}\left(\mathrm{Spec}(K(x,y,z)), \mu_2^{\otimes 3}\right) \to H^3_{et}\left(\mathrm{Spec}(k(X)), \mu_2^{\otimes 3}\right)\right)$$

$$\|$$

$$\langle [a_1] \cup [a_2] \cup [a_3] \rangle$$

where $[a_i] \in k(x,y,z)^* / (k(x,y,z)^*)^2 \cong H^1_{et}\left(\mathrm{Spec}(k(x,y,z)), \mu_2\right)$ and "$\cup$" is the cup product in ètale cohomology.

This theorem is used to relate unramified cohomology of X w/ unramified cohomology of $\mathbb{P}^3$, which is easier to handle.

Thm (CT-01)

Let X be the quadric bundle on $\mathbb{P}^3$ defined before, w/ $a_1 = f_1$, $a_2 = f_2$, $a_3 = g_1 g_2$ satisfying

(1) $f_1 = -\frac{q(x,y,z)}{p(x,y,z)}$, $\deg p, q \leq 2$

(2) $\exists D \subseteq \mathbb{P}^3$, $\mathrm{res}_D([f_1] \cup [f_2] \cup [g_1]) \neq 0$

(3) $\forall D \subseteq \mathbb{P}^3$, either $\mathrm{res}_D([f_1] \cup [f_2] \cup [g_1]) = 0$
or $\mathrm{res}_D([f_1] \cup [f_2] \cup [g_2]) = 0$

computations in $H^i(k(\mathbb{P}^3))$ are easier!

then:

(1) X unirational (w/ $\mathrm{Br}(X) = 0$)

(2) $\pi^*([f_1] \cup [f_2] \cup [g_1]) \neq 0$ in $H^3(k(X), \mu_2^{\otimes 3})$
and $\mathrm{res}_D\, \pi^*([f_1] \cup [f_2] \cup [g_1]) = 0$ $\forall D \subseteq X$

$$\Rightarrow H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}) \neq 0$$

<u>In particular</u>

let $\ell_i := \varepsilon_x x + \varepsilon_y y + \varepsilon_z z + \varepsilon_t t \qquad \varepsilon_* \in \{0,1\}$

(in this way we have 16 linear forms)

$f_1 = \frac{x}{y}, \quad f_2 = \frac{z}{t}, \quad g_1 = \prod(\ell + \ell_i)/t^{16}$ where ℓ linear form

$g_2 = \prod(m + \ell_i)/t^{16}$ where m lin form

$$\Rightarrow \left\{ X_0^2 + \frac{x}{y} X_1^2 + \frac{z}{t} X_2^2 + \frac{xz}{yt} X_3^2 + g_1 g_2 X_4^2 = 0 \right\}$$ is CT-O example!